HOMOLOGY OF SL_n AND GL_n OVER AN INFINITE FIELD

B. MIRZAII

ABSTRACT. The homology of $\mathrm{GL}_n(F)$ and $\mathrm{SL}_n(F)$ is studied, where F is an infinite field. Our main theorem states that the natural map $H_4(\mathrm{GL}_3(F),k) \to H_4(\mathrm{GL}_4(F),k)$ is injective where k is a field with $\mathrm{char}(k) \neq 2,3$. For algebraically closed field F, we prove a better result, namely, $H_4(\mathrm{GL}_3(F),\mathbb{Z}) \to H_4(\mathrm{GL}_4(F),\mathbb{Z})$ is injective. We will prove a similar result replacing GL by SL. This is used to investigate the indecomposable part of the K-group $K_4(F)$.

1. INTRODUCTION

In the beginning of the 1970's two type of K-groups in algebra appeared: Quillen's K-groups and Milnor's K-groups. For a field F, Quillen defined the K-group $K_n(F)$ as the n-th homotopy group of the space $B\mathrm{GL}(F)^+$ and Milnor defined the K-group $K_n^M(F)$ as the n-th degree part of $T(F^*)/\langle a\otimes (1-a) : a \in F^* - \{1\}\rangle$, where $T(F^*) := \mathbb{Z} \oplus F^* \oplus F^* \otimes F^* \oplus \cdots$ is the tensor algebra of F^*. There is a canonical ring homomorphism $K_*^M(F) \to K_*(F)$, therefore a canonical homomorphism $K_n^M(F) \to K_n(F)$. On the other hand, the Hurewicz theorem, in algebraic topology, relates homotopy groups to homology groups, which are much easier to calculate. This in turn provides a homomorphism from $K_n(F)$ to the n-th integral homology of the stable group $\mathrm{GL}(F)$.

One of the important approaches to investigate K-groups is by means of their relation with integral homology groups of $\mathrm{GL}(F)$ and Milnor K-groups. Suslin's stability theorem states that for an infinite field F the natural map $H_i(\mathrm{GL}_n(F),\mathbb{Z}) \to H_i(\mathrm{GL}(F),\mathbb{Z})$ is bijective if $n \geq i$ [12]. Using this Suslin constructed a map from $H_n(\mathrm{GL}_n(F),\mathbb{Z})$ to Milnor's K_n-group $K_n^M(F)$, denoted by s_n, such that the sequence

$$H_n(\mathrm{GL}_{n-1}(F),\mathbb{Z}) \xrightarrow{H_n(\mathrm{inc})} H_n(\mathrm{GL}_n(F),\mathbb{Z}) \xrightarrow{s_n} K_n^M(F) \longrightarrow 0$$

is exact. Combining these two results he constructed a map from $K_n(F)$ to $K_n^M(F)$ such that the composite homomorphism

$$K_n^M(F) \to K_n(F) \to K_n^M(F)$$

coincides with the multiplication by $(-1)^{n-1}(n-1)!$ [12, Sec. 4].

Now one might ask about the kernel of $H_n(\mathrm{inc})$ in the above exact sequence. In this direction, Suslin posed a problem, which is now referred to as 'a conjecture by Suslin' (see [10, 4.13] and [3, 7.7]).

Injectivity Conjecture. *For any infinite field F the natural homomorphism*
$$H_n(\mathrm{inc}) : H_n(\mathrm{GL}_{n-1}(F), \mathbb{Q}) \to H_n(\mathrm{GL}_n(F), \mathbb{Q})$$
is injective.

This conjecture is easy if $n = 1, 2$. For $n = 3$ the conjecture was proved positively by Sah [10] and Elbaz-Vincent [6]. The conjecture is proven in full for number fields [3]. The proof of this conjecture for $n = 4$ is the main goal of this paper (Theorem 5.6).

Here we take a general step towards this conjecture. We show that up to an induction step, the above conjecture follows from the exactness of a certain complex. Let
$$P_n := H_n(inc) : H_n(\mathrm{GL}_{n-1}(F), \mathbb{Q}) \to H_n(\mathrm{GL}_n(F), \mathbb{Q})$$
and let Q_n be the complex
$$H_n(F^{*2} \times \mathrm{GL}_{n-2}(F), \mathbb{Q}) \overset{\beta_2^{(n)}}{\to} H_n(F^* \times \mathrm{GL}_{n-1}(F), \mathbb{Q}) \overset{\beta_1^{(n)}}{\to} H_n(\mathrm{GL}_n(F), \mathbb{Q}) \to 0,$$
where $\beta_1^{(n)} = H_n(\mathrm{inc})$ and $\beta_2^{(n)} = H_n(\alpha) - H_n(\mathrm{inc})$,
$$\alpha : F^{*2} \times \mathrm{GL}_{n-2}(F) \to F^* \times \mathrm{GL}_{n-1}(F), \quad \mathrm{diag}(a, b, A) \to \mathrm{diag}(b, a, A).$$

Proposition. *Let F be an infinite field. If Q_n is exact and if P_{n-2} and P_{n-1} are injective, then P_n is injective.*

The proof follows from a careful analysis of some spectral sequences, connecting the homology of the groups $F^{*p} \times \mathrm{GL}_{n-p}(F)$ for different values of p. One of the main ingredient in the proof of this proposition is a construction of an explicit map from $K_n^M(F)$ to $H_n(\mathrm{GL}_n(F), \mathbb{Z})$, denoted by ν_n (this was only known for $n = 2$). This construction fits in our previous theory, that is, the composite homomorphism
$$K_n^M(F) \xrightarrow{\nu_n} H_n(\mathrm{GL}_n(F), \mathbb{Z}) \xrightarrow{s_n} K_n^M(F)$$
coincides with the multiplication by $(-1)^{n-1}(n-1)!$. As we mentioned above, there is a homomorphism from $K_n^M(F)$ to $H_n(\mathrm{GL}_n(F), \mathbb{Z})$ that factors through $K_n(F)$, denoted by h_n. We don't know whether these two maps coincide.

Here are our main results.

Theorem. *Let F be an infinite field.*

(i) *The complex*
$$H_4(F^{*2} \times \mathrm{GL}_2(F), \mathbb{Z}) \overset{\beta_2^{(4)}}{\to} H_4(F^* \times \mathrm{GL}_3(F), \mathbb{Z}) \overset{\beta_1^{(4)}}{\to} H_4(\mathrm{GL}_4(F), \mathbb{Z}) \to 0$$
is exact.

(ii) *Let k be a field with* $\mathrm{char}(k) \neq 2, 3$. *Then*
$$H_4(\mathrm{inc}) : H_4(\mathrm{GL}_3(F), k) \to H_4(\mathrm{GL}_4(F), k)$$

is injective.

(iii) *If F is algebraically closed, then*

$$H_4(\mathrm{inc}) : H_4(\mathrm{GL}_3(F), \mathbb{Z}) \to H_4(\mathrm{GL}_4(F), \mathbb{Z})$$

is injective.

Also we show that a similar result as in part (ii) and (iii) of the above theorem is true if we replace GL with SL. As an application we will study the indecomposable part of $K_4(F)$. Namely we will prove that for an algebraically closed field F, $K_4(F)^{\mathrm{ind}} := \mathrm{coker}(K_4^M(F) \to K_4(F))$ embeds in $H_4(\mathrm{SL}_3(F), \mathbb{Z})$.

Here we establish some notation. In this note, by $H_i(G)$ we mean the i-th integral homology of the group G. We use the bar resolution to define the homology of a group [4, Chap. I, Section 5]. Define $\mathbf{c}(g_1, g_2, \dots, g_n) = \sum_{\sigma \in \Sigma_n} \mathrm{sign}(\sigma)[g_{\sigma(1)}|g_{\sigma(2)}|\dots|g_{\sigma(n)}] \in H_n(G)$, where $g_i \in G$ pairwise commute and Σ_n is the symmetric group of degree n. By GL_n and SL_n we mean $\mathrm{GL}_n(F)$ and $\mathrm{SL}_n(F)$, where F is an infinite field. Note that GL_0 is the trivial group and $\mathrm{GL}_1 = F^*$. By F^{*m} we mean $F^* \times \cdots \times F^*$ (m-times) or the subgroup of F^*, $\{a^m | a \in F^*\}$, depending on the context. This shall not cause any confusion. The i-th factor of $F^{*m} = F^* \times \cdots \times F^*$, ($m$-times), is denoted by F_i^*.

2. Homology of SL_n

The action of F^* on SL_n defined by $a.A := \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} A \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ induces an action of F^* on $H_i(\mathrm{SL}_n)$. In this article by $H_i(\mathrm{SL}_n)_{F^*}$ we simply mean $H_0(F^*, H_i(\mathrm{SL}_n))$. It is easy to see that the natural map $\mathrm{SL}_n \to \mathrm{SL}$ induces a map of homology groups $H_i(\mathrm{SL}_n)_{F^*} \to H_i(\mathrm{SL})$.

The map $\delta : \mathrm{GL} \to \mathrm{SL}$ given by $M \mapsto \begin{pmatrix} \det(M)^{-1} & 0 \\ 0 & M \end{pmatrix}$ induces a homomorphism $H_i(\delta) : H_i(\mathrm{GL}) \to H_i(\mathrm{SL})$ such that the composition $H_i(\mathrm{SL}) \xrightarrow{H_i(\mathrm{inc})} H_i(\mathrm{GL}) \xrightarrow{H_i(\delta)} H_i(\mathrm{SL})$ is the identity map. Therefore $H_i(\mathrm{SL})$ embeds in $H_i(\mathrm{GL})$.

Lemma 2.1. (i) *$H_i(\mathrm{SL}_n)_{F^*} \simeq H_i(\mathrm{SL})$ for $n \geq i$. In particular if F is algebraically closed, then $H_i(\mathrm{SL}_n) \simeq H_i(\mathrm{SL})$ for $n \geq i$.*

(ii) *Let $A := \mathbb{Z}[\frac{1}{n-1}]$ and let $q \leq n-1$. Then for $p \geq 0$ the map*

$$H_p(F^*, H_q(\mathrm{SL}_{n-1}, A)) \to H_p(F^*, H_q(\mathrm{SL}, A)),$$

induced by the map of pair $(\mathrm{id}, \mathrm{inc}) : (F^*, \mathrm{SL}_{n-1}) \to (F^*, \mathrm{SL})$, is isomorphism.

Proof. The part (i) is rather well known (see [10, 2.7]).

(ii) If $q = 0$, then the claim is trivial. So let $q \geq 1$. The short exact sequence

$$1 \to \mu_{n-1,F} \to F^* \xrightarrow{(.)^{n-1}} F^{*n-1} \to 1$$

gives us the Lyndon-Hochschild-Serre spectral sequence

$$\mathcal{E}^2_{r,s} = H_r(F^{*n-1}, H_s(\mu_{n-1,F}, T)) \Rightarrow H_{r+s}(F^*, T),$$

where $T = H_q(\mathrm{SL}_{n-1}, A)$. Since the order of $\mu_{n-1,F}$ is invertible in A, $H_s(\mu_{n-1,F}, T) = 0$ for $s \geq 1$ [4, 10.1]. Thus $\mathcal{E}^2_{r,s} = 0$ for $s \geq 1$. The action of F^{*n-1} on $\mu_{n-1,F}$ and T is trivial, so $\mathcal{E}^2_{0,0} = H_0(\mu_{n-1,F}, T) = H_0(\mu_{n-1,F}, H_q(\mathrm{SL}_{n-1}, A))$. From this and (i) one deduces that

$$\mathcal{E}^\infty_{0,0} \simeq H_0(\mu_{n-1,F}, H_q(\mathrm{SL}_{n-1}, A)) \simeq H_0(F^*, H_q(\mathrm{SL}_{n-1}, A)) \simeq H_q(\mathrm{SL}, A)$$

and therefore $\mathcal{E}^\infty_{r,0} \simeq \mathcal{E}^2_{r,0} = H_r(F^{*n-1}, H_q(\mathrm{SL}, A))$. An easy analysis shows that

$$H_r(F^{*n-1}, H_q(\mathrm{SL}, A)) \simeq H_r(F^*, H_q(\mathrm{SL}_{n-1}, A)).$$

Once more from the short exact sequence $1 \to F^{*n-1} \to F^* \to F^*/F^{*n-1} \to 1$ one gets the Lyndon-Hochschild-Serre spectral sequence

$$\mathcal{E}'^2_{r,s} = H_r(F^*/F^{*n-1}, H_s(F^{*n-1}, S)) \Rightarrow H_{r+s}(F^*, S)$$

where $S = H_q(\mathrm{SL}, A)$. It is easy to see that $\mathcal{E}'^2_{r,s} = 0$ for $r \geq 1$ and

$$\mathcal{E}'^\infty_{0,s} \simeq \mathcal{E}'^2_{0,s} = H_0(F^*/F^{*n-1}, H_s(F^{*n-1}, S)) = H_s(F^{*n-1}, H_q(\mathrm{SL}, A)).$$

This implies that $H_s(F^{*n-1}, H_q(\mathrm{SL}, A)) \simeq H_s(F^*, H_q(\mathrm{SL}, A))$. Hence for $r \geq 0$ and $q \leq n-1$,

$$H_r(F^*, H_q(\mathrm{SL}_{n-1}, A)) \simeq H_r(F^*, H_q(\mathrm{SL}, A)).$$

It is not difficult to see that this isomorphism is induced by the map of pair $((\,)^{n-1}, \mathrm{inc}) : (F^*, \mathrm{SL}_{n-1}) \to (F^*, \mathrm{SL})$. In a similar way, one can prove that the map of pair $((\,)^{n-1}, \mathrm{inc}) : (F^*, \mathrm{SL}) \to (F^*, \mathrm{SL})$ induces the isomorphism

$$H_r(F^*, H_q(\mathrm{SL}, A)) \simeq H_r(F^*, H_q(\mathrm{SL}, A)).$$

Applying the functor H_r to the commutative diagram

$$\begin{array}{ccc}
(F^*, \mathrm{SL}_{n-1}) & \xrightarrow{(\mathrm{id},\mathrm{inc})} & (F^*, \mathrm{SL}) \\
\downarrow{\scriptstyle (\mathrm{id},\mathrm{inc})} & & \downarrow{\scriptstyle ((\,)^{n-1},\mathrm{inc})} \\
(F^*, \mathrm{SL}_{n-1}) & \xrightarrow{((\,)^{n-1},\mathrm{inc})} & (F^*, \mathrm{SL})
\end{array}$$

gives us the isomorphism that we are looking for. □

Lemma 2.2. (i) *If* $A := \mathbb{Z}[\frac{1}{n-1}]$, *then*

$$\begin{aligned}
\mathrm{im}(H_n(\mathrm{GL}_{n-1}, A) \to H_n(\mathrm{GL}_n, A)) \cap H_n(\mathrm{SL}_n, A)_{F^*} \\
= \mathrm{im}(H_n(\mathrm{SL}_{n-1}, A)_{F^*} \to H_n(\mathrm{SL}_n, A)_{F^*}).
\end{aligned}$$

(ii) *If* F *is algebraically closed, then*

$$\mathrm{im}(H_n(\mathrm{GL}_{n-1}) \to H_n(\mathrm{GL}_n)) \cap H_n(\mathrm{SL}_n) = \mathrm{im}(H_n(\mathrm{SL}_{n-1}) \to H_n(\mathrm{SL}_n)).$$

Proof. To prove (i) consider the associated Lyndon-Hochschild-Serre spectral sequence of the diagram of extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{SL}_{n-1} & \longrightarrow & \mathrm{GL}_{n-1} & \longrightarrow & F^* & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{SL} & \longrightarrow & \mathrm{GL} & \longrightarrow & F^* & \longrightarrow & 1. \end{array}$$

From this we obtain a map of spectral sequences

$$\begin{array}{ccc} E^2_{p,q} = H_p(F^*, H_q(\mathrm{SL}_{n-1}, A)) & \Rightarrow & H_{p+q}(\mathrm{GL}_{n-1}, A) \\ \downarrow & & \downarrow \\ {E'}^2_{p,q} = H_p(F^*, H_q(\mathrm{SL}, A)) & \Rightarrow & H_{p+q}(\mathrm{GL}, A). \end{array}$$

By Lemma 2.1 for $q \leq n-1$,

$$E^2_{p,q} = H_p(F^*, H_q(\mathrm{SL}_{n-1}, A)) \simeq H_p(F^*, H_q(\mathrm{SL}, A)) = {E'}^2_{p,q},$$

which is induced by the pair $(\mathrm{id}, \mathrm{inc}) : (F^*, \mathrm{SL}_{n-1}) \to (F^*, \mathrm{SL})$.

The spectral sequences give us a map of filtrations

$$\begin{array}{ccccccccc} 0 = F_{-1} & \subseteq & F_0 & \subseteq \cdots \subseteq & F_{n-1} & \subseteq & F_n = H_n(\mathrm{GL}_{n-1}, A) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 = F'_{-1} & \subseteq & F'_0 & \subseteq \cdots \subseteq & F'_{n-1} & \subseteq & F'_n = H_n(\mathrm{GL}, A), \end{array}$$

such that $E^2_{i,n-i} \simeq F_i/F_{i-1}$, ${E'}^2_{i,n-i} \simeq F'_i/F'_{i-1}$. The commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_{i-1} & \longrightarrow & F_i & \longrightarrow & E^\infty_{i,n-i} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \wr & & \\ 0 & \longrightarrow & F'_{i-1} & \longrightarrow & F'_i & \longrightarrow & {E'}^\infty_{i,n-i} & \longrightarrow & 0 \end{array}$$

implies that $\mathrm{im}(F_i \to F'_i) \cap F'_{i-1} = \mathrm{im}(F_{i-1} \to F'_{i-1})$, for $1 \leq i \leq n-1$. From this we obtain $\mathrm{im}(F_n \to F'_n) \cap F'_0 = \mathrm{im}(F_0 \to F'_0)$. Since $H_n(\mathrm{SL}, A) \to H_n(\mathrm{GL}, A)$ is injective, $F'_0 = {E'}^\infty_{0,n} = H_n(\mathrm{SL}, A)$. In a similar way one obtains $F_0 = E^\infty_{0,n} = H_n(\mathrm{SL}_{n-1}, A)_{F^*}/T$. It is easy to see that the map $F_0 \to F'_0$ is induced by the natural map $\mathrm{SL}_{n-1} \to \mathrm{SL}$ and so

$$T \subseteq \ker(H_n(\mathrm{SL}_{n-1}, A)_{F^*} \to H_n(\mathrm{SL}, A)).$$

This completes the proof of (i). The proof of (ii) is analogue to (i). □

Theorem 2.3. (i) *If* $B := \mathbb{Z}[\frac{1}{(n-1)!}]$, *then*

$$H_n(\mathrm{SL}_{n-1}, B)_{F^*} \to H_n(\mathrm{SL}_n, B)_{F^*} \to K_n^M(F) \otimes B \to 0$$

is exact.

(ii) *If* F *is algebraically closed, then*

$$H_n(\mathrm{SL}_{n-1}) \to H_n(\mathrm{SL}_n) \to K_n^M(F) \to 0$$

is exact.

Proof. (i) Consider the commutative diagram

$$\begin{array}{ccccccc} H_n(\mathrm{SL}_{n-1}, B)_{F^*} & \longrightarrow & H_n(\mathrm{SL}_n, B)_{F^*} & \longrightarrow & K_n^M(F) \otimes B & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ H_n(\mathrm{GL}_{n-1}, B) & \longrightarrow & H_n(\mathrm{GL}_n, B) & \longrightarrow & K_n^M(F) \otimes B & \longrightarrow & 0. \end{array}$$

Note that the second row in the above diagram is exact. Suslin's map $s_n : H_n(\mathrm{GL}_n) \to K_n^M(F)$ has the property that the composition $K_n^M(F) \to K_n(F) \to H_n(\mathrm{GL}_n) \to K_n^M(F)$ coincides with multiplication by $(-1)^{n-1}(n-1)!$. The Hurewicz map $K_n(F) \to H_n(\mathrm{GL}_n)$ factors through $H_n(\mathrm{SL}_n)_{F^*} \simeq H_n(\mathrm{SL})$. Since $(n-1)!$ is invertible in $B = \mathbb{Z}[\frac{1}{(n-1)!}]$, the restriction map $s_n : H_n(\mathrm{SL}_n, B)_{F^*} \to K_n^M(F) \otimes B$ is surjective. The exactness of the complex follows from the exactness of the second row of the above diagram and Lemma 2.2.

The proof of (ii) is analogue. In this case one should use the fact that for an algebraically closed field F, $K_n^M(F)$ is uniquely divisible [2, 1.2]. □

Let $K_n^M(F) \to K_n(F)$ be the natural map from the Milnor K-group to the Quillen K-group. Define $K_n(F)^{\mathrm{ind}} := \mathrm{coker}(K_n^M(F) \to K_n(F))$. This group is called the indecomposable part of $K_n(F)$.

Corollary 2.4. *Let k be a field such that $(n-1)! \in k^*$. Assume $H_n(\mathrm{inc}) : H_n(\mathrm{GL}_{n-1}, k) \to H_n(\mathrm{GL}_n, k)$ is injective. Then*

(i) *$H_n(\mathrm{SL}_{n-1}, k)_{F^*} \xrightarrow{H_n(\mathrm{inc})} H_n(\mathrm{SL}_n, k)_{F^*}$ is injective.*

(ii) *$K_n(F)^{\mathrm{ind}} \otimes k$ embeds in $H_n(\mathrm{SL}_{n-1}, k)_{F^*}$ and*

$$H_n(\mathrm{SL}_{n-1}, k)_{F^*}/K_n(F)^{\mathrm{ind}} \otimes k \simeq H_n(\mathrm{SL}_n, k)_{F^*}/K_n(F) \otimes k.$$

(iii) *In $H_n(\mathrm{GL}_n, k)$, $K_n(F) \otimes k \cap H_n(\mathrm{GL}_{n-1}, k)$ coincides with*

$$K_n(F) \otimes k \cap H_n(\mathrm{SL}_{n-1}, k)_{F^*} = K_n(F)^{\mathrm{ind}} \otimes k.$$

Proof. (i) This follows from the assumptions, 2.3, the commutativity of the diagram in the proof of 2.3 and the injectivity of the natural map $\mathrm{inc}_* : H_n(\mathrm{SL}_{n-1}, k)_{F^*} \to H_n(\mathrm{GL}_{n-1}, k)$. For the proof of the injectivity of inc_* consider the map $\gamma_{n-1} : F^* \times \mathrm{SL}_{n-1} \to \mathrm{GL}_{n-1}$ given by $(a, A) \mapsto aA$. By tracing the kernel and cokernel of this map we obtain the exact sequence

$$1 \to \mu_{n-1,F} \to F^* \times \mathrm{SL}_{n-1} \stackrel{\gamma_{n-1}}{\to} \mathrm{GL}_{n-1} \to F^*/F^{*n-1} \to 1.$$

From this we get two short exact sequences

$$1 \to \mu_{n-1,F} \to F^* \times \mathrm{SL}_{n-1} \to \mathrm{im}(\gamma_{n-1}) \to 1,$$
$$1 \to \mathrm{im}(\gamma_{n-1}) \to \mathrm{GL}_{n-1} \to F^*/F^{*n-1} \to 1.$$

Writing the Lyndon-Hochschild-Serre spectral sequence of the above short exact sequences (with coefficient in k) and carrying out an easy analysis, one gets

$$H_n(F^* \times \mathrm{SL}_{n-1}, k)_{F^*} \simeq H_n(\mathrm{GL}_{n-1}, k).$$

Now by the Künneth theorem $H_n(\mathrm{SL}_{n-1}, k)_{F^*}$ embeds in $H_n(\mathrm{GL}_{n-1}, k)$, which is induced by the natural map $\mathrm{inc} : \mathrm{SL}_{n-1} \to \mathrm{GL}_{n-1}$.

(ii) By (i) the complex

$$0 \to H_n(\mathrm{SL}_{n-1}, k)_{F^*} \to H_n(\mathrm{SL}_n, k)_{F^*} \to K_n^M(F) \otimes k \to 0$$

is exact. By Suslin's construction of a map $K_n(F) \to K_n^M(F)$ [12, Section 4], we have a splitting exact sequence

$$0 \to K_n(F)^{\mathrm{ind}} \otimes k \to K_n(F) \otimes k \to K_n^M(F) \otimes k \to 0.$$

The claims follow from applying the Snake lemma to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_n(F)^{\mathrm{ind}} \otimes k & \longrightarrow & K_n(F) \otimes k & \longrightarrow & K_n^M(F) \otimes k & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g_n} & & \downarrow{\scriptstyle h_n} & & \downarrow & & \\ 0 & \longrightarrow & H_n(\mathrm{SL}_{n-1}, k)_{F^*} & \longrightarrow & H_n(\mathrm{SL}_n, k)_{F^*} & \longrightarrow & K_n^M(F) \otimes k & \longrightarrow & 0, \end{array}$$

where h_n is the Hurewicz map $K_n(F) \otimes k \to H_n(\mathrm{SL}, k) \simeq H_n(\mathrm{SL}_n, k)_{F^*}$ which is injective [14, Prop. 3, p. 507] and $g_n = h_n|_{K_n(F)^{\mathrm{ind}}}$. □

3. Milnor K-groups

We start this section with an easy lemma.

Lemma 3.1. *Let G be a group and let $g_1, g_2, h_1, \dots, h_n \in G$ such that each pair commute. Let $C_G(\langle h_1, \dots, h_n \rangle)$ be the subgroup of G consisting of all elements of G that commute with all h_i, $i = 1, \dots, n$. If $\mathbf{c}(g_1, g_2) = 0$ in $H_2(C_G(\langle h_1, \dots, h_n \rangle))$, then $\mathbf{c}(g_1, g_2, h_1, \dots, h_n) = 0$ in $H_{n+2}(G)$.*

Proof. The homomorphism $C_G(\langle h_1, \dots, h_n \rangle) \times \langle h_1, \dots, h_n \rangle \to G$ defined by $(g, h) \to gh$ induces the map

$$H_2(C_G(\langle h_1, \dots, h_n \rangle)) \otimes H_n(\langle h_1, \dots, h_n \rangle) \to H_{n+2}(G).$$

The claim follows from the fact that $\mathbf{c}(g_1, g_2, h_1, \dots, h_n)$ is the image of $\mathbf{c}(g_1, g_2) \otimes \mathbf{c}(h_1, \dots, h_n)$ under this map. □

Definition 3.2. Let $A_{i,n} := \mathrm{diag}(a_i, \dots, a_i, a_i^{-(i-1)}, I_{n-i}) \in \mathrm{GL}_n$. We define $[a_1, \dots, a_n] := \mathbf{c}(A_{1,n}, \dots, A_{n,n}) \in H_n(\mathrm{GL}_n)$.

Proposition 3.3. (i) *The map $\nu_n : K_n^M(F) \to H_n(\mathrm{GL}_n)$ defined by*

$$\{a_1, \dots, a_n\} \mapsto [a_1, \dots, a_n]$$

is a homomorphism.

(ii) *Let $s_n : H_n(\mathrm{GL}_n) \to K_n^M(F)$ be the map defined by Suslin. Then $s_n \circ \nu_n$ coincides with the multiplication by $(-1)^{(n-1)}(n-1)!$.*

Proof. (i) It is well know that the Hurewicz map $h_2 : K_2^M(F) \simeq K_2(F) \to H_2(\mathrm{GL})$ is defined by $\{a, b\} \mapsto \mathbf{c}(\mathrm{diag}(a, 1, a^{-1}), \mathrm{diag}(b, b^{-1}, 1))$. It is easy to see that in $H_2(\mathrm{GL})$, $[a, b] = \mathbf{c}(\mathrm{diag}(a, 1, a^{-1}), \mathrm{diag}(b, b^{-1}, 1))$. The stability isomorphism $H_2(\mathrm{GL}_2) \simeq H_2(\mathrm{GL})$ implies that the map $K_2^M(F) = K_2(F) \to H_2(\mathrm{GL}_2)$, $\{a, b\} \mapsto [a, b]$ is well defined. Now by lemma 3.1, $[a_1, 1 - a_1, a_3, \dots, a_n] = 0$. To complete the proof of (i) it is sufficient to prove that

$$[a_1, \dots, a_{n-2}, a_{n-1}, a_n] = -[a_1, \dots, a_{n-2}, a_n, a_{n-1}].$$

This can be done in the following way;

$$
\begin{aligned}
&[a_1,\dots,a_{n-2},a_{n-1},a_n]\\
&=\mathbf{c}(A_{1,n},\dots,A_{n-2,n},A_{n-1,n},A_{n,n})\\
&=\mathbf{c}(A_{1,n},\dots,A_{n-2,n},\mathrm{diag}(a_{n-1}I_{n-2},a_{n-1},a_{n-1}^{-(n-1)}),A_{n,n})\\
&+\mathbf{c}(A_{1,n},\dots,A_{n-2,n},\mathrm{diag}(I_{n-2},a_{n-1}^{-(n-1)},a_{n-1}^{(n-1)}),A_{n,n})\\
&=\mathbf{c}(A_{1,n},\dots,A_{n-2,n},\mathrm{diag}(a_{n-1}I_{n-2},a_{n-1},a_{n-1}^{-(n-1)}),\\
&\qquad\qquad\qquad\qquad\mathrm{diag}(a_nI_{n-2},a_n^{-(n-2)},1))\\
&+\mathbf{c}(A_{1,n},\dots,A_{n-2,n},\mathrm{diag}(a_{n-1}I_{n-2},a_{n-1},a_{n-1}^{-(n-1)}),\\
&\qquad\qquad\qquad\qquad\mathrm{diag}(I_{n-2},a_n^{(n-1)},a_n^{-(n-1)}))\\
&+\mathbf{c}(A_{1,n},\dots,A_{n-2,n},\mathrm{diag}(I_{n-2},a_{n-1}^{-(n-1)},a_{n-1}^{(n-1)}),A_{n,n})\\
&=-[a_1,\dots,a_{n-2},a_n,a_{n-1}]\\
&+\mathbf{c}(A_{1,n},\dots,A_{n-2,n},\mathrm{diag}(I_{n-2},a_{n-1},a_{n-1}^{-(n-1)}),\\
&\qquad\qquad\qquad\qquad\mathrm{diag}(I_{n-2},a_n^{(n-1)},a_n^{-(n-1)}))\\
&+\mathbf{c}(A_{1,n},\dots,A_{n-2,n},\mathrm{diag}(a_{n-1}I_{n-2},1,1),\mathrm{diag}(I_{n-2},a_n^{(n-1)},a_n^{-(n-1)}))\\
&+\mathbf{c}(A_{1,n},\dots,A_{n-2,n},\mathrm{diag}(I_{n-2},a_{n-1}^{-(n-1)},a_{n-1}^{(n-1)}),\mathrm{diag}(I_{n-2},a_n,a_n^{-(n-1)}))\\
&+\mathbf{c}(A_{1,n},\dots,A_{n-2,n},\mathrm{diag}(I_{n-2},a_{n-1}^{-(n-1)},a_{n-1}^{(n-1)}),\mathrm{diag}(a_nI_{n-2},1,1))\\
&=-[a_1,\dots,a_n,a_{n-1}].
\end{aligned}
$$

(ii) Let τ_n be the composite map $K_n^M(F)\to K_n(F)\xrightarrow{h_n}H_n(\mathrm{GL}_n)$. Then $s_n\circ\tau_n$ coincides with the multiplication by $(-1)^{(n-1)}(n-1)!$ [12, section 4]. It is well known that the composite map $K_n^M(F)\xrightarrow{\tau_n}H_n(\mathrm{GL}_n)\to H_n(\mathrm{GL}_n)/H_n(\mathrm{GL}_{n-1})$ is an isomorphism and it is defined by $\{a_1,\dots,a_n\}\mapsto(a_1\cup\cdots\cup a_n)\mod H_n(\mathrm{GL}_{n-1})$, where

$$a_1\cup a_2\cup\cdots\cup a_n=\mathbf{c}(\mathrm{diag}(a_1,I_{n-1}),\mathrm{diag}(1,a_2,I_{n-2}),\dots,\mathrm{diag}(I_{n-1},a_n))$$

(see [9, Remark 3.27]). Also we know that s_n factors as

$$H_n(\mathrm{GL}_n)\to H_n(\mathrm{GL}_n)/H_n(\mathrm{GL}_{n-1})\to K_n^M(F).$$

Our claim follows from the fact that modulo $H_n(\mathrm{GL}_{n-1})$

$$[a_1,\dots,a_n]=(-1)^{n-1}(n-1)!(a_1\cup\cdots\cup a_n).$$

□

4. Homology of GL_n

Let $C_l(F^n)$ and $D_l(F^n)$ be the free abelian groups with a basis consisting of $(\langle v_0\rangle,\dots,\langle v_l\rangle)$ and $(\langle w_0\rangle,\dots,\langle w_l\rangle)$ respectively, where every $\min\{l+1,n\}$ of $v_i\in F^n$ and every $\min\{l+1,2\}$ of $w_i\in F^n$ are linearly independent. By $\langle v_i\rangle$ we mean the line passing through vectors v_i and 0. Let $\partial_0:C_0(F^n)\to C_{-1}(F^n):=\mathbb{Z}$, $\sum_i n_i(\langle v_i\rangle)\mapsto\sum_i n_i$ and $\partial_l=\sum_{i=0}^l(-1)^i d_i:C_l(F^n)\to C_{l-1}(F^n)$, $l\geq 1$, where

$$d_i((\langle v_0\rangle,\dots,\langle v_l\rangle))=(\langle v_0\rangle,\dots,\widehat{\langle v_i\rangle},\dots,\langle v_l\rangle).$$

Define the differential $\tilde{\partial}_l = \sum_{i=0}^{l}(-1)^i \tilde{d}_i : D_l(F^n) \to D_{l-1}(F^n)$ similar to ∂_l. Set $L_0 = \mathbb{Z}$, $M_0 = \mathbb{Z}$, $L_l = C_{l-1}(F^n)$ and $M_l = D_{l-1}(F^n)$, $l \geq 1$. It is easy to see that the complexes

$$\begin{array}{rl} L_* : & 0 \leftarrow L_0 \leftarrow L_1 \leftarrow \cdots \leftarrow L_l \leftarrow \cdots \\ M_* : & 0 \leftarrow M_0 \leftarrow M_1 \leftarrow \cdots \leftarrow M_l \leftarrow \cdots \end{array}$$

are exact. Take a GL_n-resolution $P_* \to \mathbb{Z}$ of $\mathbb{Z}$ with trivial GL_n-action. From the double complexes $L_* \otimes_{\mathrm{GL}_n} P_*$ and $M_* \otimes_{\mathrm{GL}_n} P_*$ we obtain two first quadrant spectral sequences converging to zero with

$$E^1_{p,q}(n) = \begin{cases} H_q(F^{*p} \times \mathrm{GL}_{n-p}) & \text{if } 0 \leq p \leq n \\ H_q(\mathrm{GL}_n, C_{p-1}(F^n) & \text{if } p \geq n+1, \end{cases}$$

$$\tilde{E}^1_{p,q}(n) = \begin{cases} H_q(F^{*p} \times \mathrm{GL}_{n-p}) & \text{if } 0 \leq p \leq 2 \\ H_q(\mathrm{GL}_n, D_{p-1}(F^n)) & \text{if } p \geq 3. \end{cases}$$

For $1 \leq p \leq n$, and $q \geq 0$, $d^1_{p,q}(n) = \sum_{i=1}^{p}(-1)^{i+1} H_q(\alpha_{i,p})$, where

$$\begin{array}{l} \alpha_{i,p} : F^{*p} \times \mathrm{GL}_{n-p} \to F^{*p-1} \times \mathrm{GL}_{n-p+1}, \\ \mathrm{diag}(a_1, \dots, a_p, A) \mapsto \mathrm{diag}(a_1, \dots, \widehat{a_i}, \dots, a_p, \begin{pmatrix} a_i & 0 \\ 0 & A \end{pmatrix}). \end{array}$$

In particular for $0 \leq p \leq n$, $d^1_{p,0}(n) = \begin{cases} \mathrm{id}_{\mathbb{Z}} & \text{if } p \text{ is odd} \\ 0 & \text{if } p \text{ is even} \end{cases}$, so $E^2_{p,0}(n) = 0$ for $p \leq n-1$. It is also easy to see that $E^2_{n,0}(n) = E^2_{n+1,0}(n) = 0$. See the proof of [7, Thm. 3.5] for more details.

Let k be a field and $C'_i(F^n) := C_i(F^n) \otimes k$. Consider the commutative diagram of complexes

$$\begin{array}{ccccccccccc} 0 & \leftarrow & k & \leftarrow & C'_0(F^n) & \leftarrow & C'_1(F^n) & \leftarrow & C'_2(F^n) & \leftarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \leftarrow & 0 & \leftarrow & C'_0(F^n) & \leftarrow & C'_1(F^n) & \leftarrow & C'_2(F^n) & \leftarrow & \cdots, \end{array}$$

where the first vertical map is zero and the other vertical maps are just identity maps. This gives a map of the first quadrant spectral sequences

$$E^1_{p,q}(n) \otimes k \to \mathcal{E}^1_{p,q}(n),$$

where $\mathcal{E}^1_{p,q}(n) \Rightarrow H_{p+q-1}(\mathrm{GL}_n, k)$ with $\mathcal{E}^1$-terms

$$\mathcal{E}^1_{p,q}(n) = \begin{cases} E^1_{p,q}(n) \otimes k & \text{if } p \geq 1 \\ 0 & \text{if } p = 0 \end{cases}$$

and differentials $\mathfrak{d}^1_{p,q}(n) = \begin{cases} d^1_{p,q}(n) \otimes \mathrm{id}_k & \text{if } p \geq 2 \\ 0 & \text{if } p = 1 \end{cases}$. It is not difficult to see that $E^\infty_{p,q} \otimes k = \mathcal{E}^\infty_{p,q}$ if $p \neq 1$, $q \leq n$ and $p+q \leq n+1$. Hence $\mathcal{E}^\infty_{p,q} = 0$ if $p \neq 1$, $q \leq n$ and $p+q \leq n+1$.

We look at the second spectral sequence in a different way. The complex

$$0 \leftarrow C'_0(F^n) \leftarrow C'_1(F^n) \leftarrow \cdots \leftarrow C'_l(F^n) \leftarrow \cdots$$

induces a first quadrant spectral sequence ${\mathcal{E}'}^1_{p,q}(n) \Rightarrow H_{p+q}(\mathrm{GL}_n, k)$, where ${\mathcal{E}'}^1_{p,q}(n) = \mathcal{E}^1_{p+1,q}(n)$ and ${\mathfrak{d}'}^1_{p,q}(n) = \mathfrak{d}^1_{p+1,q}(n)$. Thus ${\mathcal{E}'}^\infty_{p,q}(n) = 0$ if $p \geq 1$, $q \leq n-1$ and $p+q \leq n$.

Proposition 4.1. *Let $n \geq 3$ and let k be a field such that $(n-1)! \in k^*$. Let the complex*

$$H_n(F^{*2} \times \mathrm{GL}_{n-2}, k) \xrightarrow{\beta_2^{(n)}} H_n(F^* \times \mathrm{GL}_{n-1}, k) \xrightarrow{\beta_1^{(n)}} H_n(\mathrm{GL}_n, k) \to 0$$

be exact, where $\beta_2^{(n)} = H_n(\alpha_{1,2}) - H_n(\alpha_{2,2})$ and $\beta_1^{(n)} = H_n(\mathrm{inc})$. If the map $H_m(\mathrm{inc}) : H_m(\mathrm{GL}_{m-1}, k) \to H_m(\mathrm{GL}_m, k)$ is injective for $m = n-1, n-2$, then $H_n(\mathrm{inc}) : H_n(\mathrm{GL}_{n-1}, k) \to H_n(\mathrm{GL}_n, k)$ is injective.

Proof. The exactness of the above complex shows that the differentials

$${\mathfrak{d}'}^r_{r,n-r+1}(n) : {\mathcal{E}'}^r_{r,n-r+1}(n) \to {\mathcal{E}'}^r_{0,n}(n)$$

are zero for $r \geq 2$. This proves that ${\mathcal{E}'}^2_{0,n}(n) \simeq {\mathcal{E}'}^\infty_{0,n}(n)$. To complete the proof it is sufficient to prove that the group $H_n(\mathrm{GL}_{n-1}, k)$ is a summand of ${\mathcal{E}'}^2_{0,n}(n)$. To prove this it is sufficient to define a map

$$\varphi : H_n(F^* \times \mathrm{GL}_{n-1}, k) \to H_n(\mathrm{GL}_{n-1}, k)$$

such that ${\mathfrak{d}'}^1_{1,n}(H_n(F^{*2} \times \mathrm{GL}_{n-2}, k)) \subseteq \ker(\varphi)$. Consider the decompositions $H_n(F^* \times \mathrm{GL}_n, k) = \bigoplus_{i=0}^n S_i$, where $S_i = H_i(F^*, k) \otimes H_{n-i}(\mathrm{GL}_{n-1}, k)$. For $2 \leq i \leq n$, the stability theorem gives the isomorphisms $H_i(F^*, k) \otimes H_{n-i}(\mathrm{GL}_{n-2}, k) \simeq S_i$. Define $\varphi : S_0 \to H_n(\mathrm{GL}_{n-1}, k)$ the identity map and for $2 \leq i \leq n$, $\varphi : S_i \simeq H_i(F^*, k) \otimes H_{n-i}(\mathrm{GL}_{n-2}, k) \to H_n(\mathrm{GL}_{n-1}, k)$ the shuffle product. To complete the definition of φ we must define it on S_1. By a theorem of Suslin [12, 3.4] and the assumption, we have the decomposition $H_{n-1}(\mathrm{GL}_{n-1}, k) \simeq H_{n-1}(\mathrm{GL}_{n-2}, k) \oplus K^M_{n-1}(F) \otimes k$. So $S_1 \simeq H_1(F^*, k) \otimes H_{n-1}(\mathrm{GL}_{n-2}, k) \oplus H_1(F^*, k) \otimes K^M_{n-1}(F) \otimes k$. Now define $\varphi : H_1(F^*, k) \otimes H_{n-1}(\mathrm{GL}_{n-2}, k) \to H_n(\mathrm{GL}_{n-1}, k)$ the shuffle product and $\varphi : H_1(F^*, k) \otimes K^M_{n-1}(F) \to H_n(\mathrm{GL}_{n-1}, k)$ the composite map

$$\begin{aligned} H_1(F^*, k) \otimes K^M_{n-1}(F) \otimes k &\xrightarrow{f} H_1(F^*, k) \otimes H_{n-1}(\mathrm{GL}_{n-1}, k) \\ &\xrightarrow{g} H_n(F^* \times \mathrm{GL}_{n-1}, k) \xrightarrow{h} H_n(\mathrm{GL}_{n-1}, k), \end{aligned}$$

where $f = \frac{1}{n-1}(\mathrm{id} \otimes \nu_{n-1})$, g is the shuffle product and h is induced by the map $F^* \times \mathrm{GL}_{n-1} \to \mathrm{GL}_{n-1}$, $\mathrm{diag}(a, A) \mapsto aA$. By the Künnuth theorem we

have the decomposition

$$\begin{aligned}
&T_0 = H_n(\mathrm{GL}_{n-2}, k),\\
&T_1 = \textstyle\bigoplus_{i=1}^{n} H_i(F_1^*, k) \otimes H_{n-i}(\mathrm{GL}_{n-2}, k),\\
&T_2 = \textstyle\bigoplus_{i=1}^{n} H_i(F_2^*, k) \otimes H_{n-i}(\mathrm{GL}_{n-2}, k),\\
&T_3 = H_1(F_1^*, k) \otimes H_1(F_2^*, k) \otimes H_{n-2}(\mathrm{GL}_{n-2}, k),\\
&T_4 = \textstyle\bigoplus_{\substack{i+j\geq 3\\ i,j\neq 0}} H_i(F_1^*, k) \otimes H_j(F_2^*, k) \otimes H_{n-i-j}(\mathrm{GL}_{n-2}, k).
\end{aligned}$$

By lemma 3.3, $T_3 = T_3' \oplus T_3''$, where

$$\begin{aligned}
&T_3' = H_1(F_1^*, k) \otimes H_1(F_2^*, k) \otimes H_{n-2}(\mathrm{GL}_{n-3}, k),\\
&T_3'' = H_1(F_1^*, k) \otimes H_1(F_2^*, k) \otimes K_{n-2}^M(F) \otimes k.
\end{aligned}$$

It is not difficult to see that ${\mathfrak{d}'}_{1,n}^{1}(T_0 \oplus T_1 \oplus T_2 \oplus T_3' \oplus T_4) \subseteq \ker(\varphi)$. Here one should use the stability theorem. To prove ${\mathfrak{d}'}_{1,n}^{1}(T_3'') \subseteq \ker(\varphi)$ we apply 3.3;

$$\begin{aligned}
&{\mathfrak{d}'}_{1,n}^{1}\big(a \otimes b \otimes \{c_1, \dots, c_{n-2}\}\big)\\
&= -\tfrac{(-1)^{n-3}}{(n-3)!}\Big(b \otimes \mathbf{c}(\mathrm{diag}(a, I_{n-2}), \mathrm{diag}(1, C_{1,n-2}), \dots, \mathrm{diag}(1, C_{n-2,n-2}))\\
&\qquad\qquad + a \otimes \mathbf{c}(\mathrm{diag}(b, I_{n-2}), \mathrm{diag}(1, C_{1,n-2}), \dots, \mathrm{diag}(1, C_{n-2,n-2}))\Big)\\
&= \tfrac{1}{(n-2)!}\Big(b \otimes [c_1, \dots, c_{n-2}, a] + a \otimes [c_1, \dots, c_{n-2}, b],\\
&\qquad\qquad - b \otimes \mathbf{c}(\mathrm{diag}(C_{1,n-2}, 1), \dots, \mathrm{diag}(C_{n-2,n-2}, 1), \mathrm{diag}(aI_{n-2}, 1))\\
&\qquad\qquad - a \otimes \mathbf{c}(\mathrm{diag}(C_{1,n-2}, 1), \dots, \mathrm{diag}(C_{n-2,n-2}, 1), \mathrm{diag}(bI_{n-2}, 1))\Big).
\end{aligned}$$

Therefore ${\mathfrak{d}'}_{1,n}^{1}\big(a \otimes b \otimes \{c_1, \dots, c_{n-2}\}\big) = (x_1, x_2) \in T_3' \oplus T_3''$, where

$$\begin{aligned}
x_1 &= -\tfrac{1}{(n-2)!}\Big(b \otimes \mathbf{c}(\mathrm{diag}(C_{1,n-2}), \dots, \mathrm{diag}(C_{n-2,n-2}), \mathrm{diag}(aI_{n-2}))\\
&\qquad\qquad + a \otimes \mathbf{c}(\mathrm{diag}(C_{1,n-2}), \dots, \mathrm{diag}(C_{n-2,n-2}), \mathrm{diag}(bI_{n-2}))\Big),\\
x_2 &= (-1)^{n-2}\Big(b \otimes \{c_1, \dots, c_{n-2}, a\} + a \otimes \{c_1, \dots, c_{n-2}, b\}\Big).
\end{aligned}$$

We have $\phi(x_1) = -\frac{1}{(n-2)!}y$, where

$$\begin{aligned}
y &= \mathbf{c}(\mathrm{diag}(b, I_{n-2}), \mathrm{diag}(1, C_{1,n-2}), \dots, \mathrm{diag}(1, C_{n-2,n-2}), \mathrm{diag}(1, aI_{n-2}))\\
&\quad + \mathbf{c}(\mathrm{diag}(a, I_{n-2}), \mathrm{diag}(1, C_{1,n-2}), \dots, \mathrm{diag}(1, C_{n-2,n-2}), \mathrm{diag}(1, bI_{n-2}))
\end{aligned}$$

and $\phi(x_2) = \frac{(-1)^{n-2}}{n-1}\frac{(-1)^{n-2}}{(n-2)!}z = \frac{1}{(n-1)!}z$, where

$$\begin{aligned}
&z =\\
&\mathbf{c}(\mathrm{diag}(bI_{n-1}), \mathrm{diag}(C_{1,n-2}, 1), \dots, \mathrm{diag}(C_{n-2,n-2}, 1), \mathrm{diag}(aI_{n-2}, a^{-(n-2)}))\\
&\mathbf{c}(\mathrm{diag}(aI_{n-1}), \mathrm{diag}(C_{1,n-2}, 1), \dots, \mathrm{diag}(C_{n-2,n-2}, 1), \mathrm{diag}(bI_{n-2}, b^{-(n-2)}))\\
&= \mathbf{c}(\mathrm{diag}(bI_{n-1}), \mathrm{diag}(C_{1,n-2}, 1), \dots, \mathrm{diag}(C_{n-2,n-2}, 1), \mathrm{diag}(aI_{n-2}, a))\\
&+ \mathbf{c}(\mathrm{diag}(bI_{n-1}), \mathrm{diag}(C_{1,n-2}, 1), \dots, \mathrm{diag}(C_{n-2,n-2}, 1), \mathrm{diag}(I_{n-2}, a^{-(n-1)}))\\
&+ \mathbf{c}(\mathrm{diag}(aI_{n-1}), \mathrm{diag}(C_{1,n-2}, 1), \dots, \mathrm{diag}(C_{n-2,n-2}, 1), \mathrm{diag}(bI_{n-2}, b))\\
&+ \mathbf{c}(\mathrm{diag}(aI_{n-1}), \mathrm{diag}(C_{1,n-2}, 1), \dots, \mathrm{diag}(C_{n-2,n-2}, 1), \mathrm{diag}(I_{n-2}, b^{-(n-1)})).
\end{aligned}$$

Hence $\phi(x_2) = \frac{-1}{(n-2)!}z'$, where

$$
\begin{aligned}
&z' = \\
&+\mathbf{c}(\mathrm{diag}(bI_{n-1}), \mathrm{diag}(C_{1,n-2}, 1), \dots, \mathrm{diag}(C_{n-2,n-2}, 1), \mathrm{diag}(I_{n-2}, a)) \\
&+\mathbf{c}(\mathrm{diag}(aI_{n-1}), \mathrm{diag}(C_{1,n-2}, 1), \dots, \mathrm{diag}(C_{n-2,n-2}, 1), \mathrm{diag}(I_{n-2}, b)) \\
&= \mathbf{c}(\mathrm{diag}(bI_{n-2}, 1), \mathrm{diag}(C_{1,n-2}, 1), \dots, \mathrm{diag}(C_{n-2,n-2}, 1), \mathrm{diag}(I_{n-2}, a)) \\
&+\mathbf{c}(\mathrm{diag}(I_{n-2}, b), \mathrm{diag}(C_{1,n-2}, 1), \dots, \mathrm{diag}(C_{n-2,n-2}, 1), \mathrm{diag}(I_{n-2}, a)) \\
&+\mathbf{c}(\mathrm{diag}(aI_{n-2}, 1), \mathrm{diag}(C_{1,n-2}, 1), \dots, \mathrm{diag}(C_{n-2,n-2}, 1), \mathrm{diag}(I_{n-2}, b)) \\
&+\mathbf{c}(\mathrm{diag}(I_{n-2}, a), \mathrm{diag}(C_{1,n-2}, 1), \dots, \mathrm{diag}(C_{n-2,n-2}, 1), \mathrm{diag}(I_{n-2}, b)) \\
&= -y.
\end{aligned}
$$

Therefore $\varphi(x_2) = \frac{-1}{(n-2)!}z' = -\frac{-1}{(n-2)!}y = -\varphi(x_1)$. This completes the proof of the fact that $\mathfrak{d}'^{1}_{1,n}(H_n(F^{*2} \times \mathrm{GL}_{n-2}, k)) \subseteq \ker(\varphi)$. □

Thus it is reasonable to conjecture

Conjecture 4.2. *Let $(n-1)! \in k^*$ and let $n \geq 3$. Then*

$$H_n(F^{*2} \times \mathrm{GL}_{n-2}, k) \xrightarrow{\beta_2^{(n)}} H_n(F^* \times \mathrm{GL}_{n-1}, k) \xrightarrow{\beta_1^{(n)}} H_n(\mathrm{GL}_n, k) \to 0,$$

is exact.

Corollary 4.3. *Let $(n-1)! \in k^*$. If Conjecture 4.2 is true for all $n \geq 3$, then $H_n(\mathrm{inc}) : H_n(\mathrm{GL}_{n-1}, k) \to H_n(\mathrm{GL}_n, k)$ is injective for all n. In particular if $k = \mathbb{Q}$, then Conjecture 4.2 implies Suslin's Injectivity Conjecture.*

Proof. This follows immediately from Proposition 4.1. □

Remark 1. (i) The surjectivity of $\beta_1^{(n)}$ is already proven by Suslin [12].

(ii) The conjecture is proven for n=3 in [8, Prop. 2. 5] and we prove it in this note for $n = 4$.

(iii) A similar result is not true for $n = 2$, that is

$$H_2(F^{*2} \times \mathrm{GL}_0) \xrightarrow{\beta_2^{(2)}} H_2(F^* \times \mathrm{GL}_1) \xrightarrow{\beta_1^{(2)}} H_2(\mathrm{GL}_2) \to 0$$

is not exact. In fact

$$\ker(\beta_1^{(2)})/\mathrm{im}(\beta_2^{(2)}) \simeq \langle x \wedge (x-1) - x \otimes (x-1) : x \in F^* \rangle$$

is a subset of $H_2(F^*) \oplus (F^* \otimes F^*)_\sigma$, where $(F^* \otimes F^*)_\sigma = (F^* \otimes F^*)/\langle a \otimes b + b \otimes a : a, b \in F^* \rangle$. To prove this let $Q(F)$ be the free abelian group with the basis $\{[x] : x \in F^* - \{1\}\}$. Denote by $\mathfrak{p}(F)$ the factor group of $Q(F)$ by the subgroup generated by the elements of the form $[x] - [y] + [y/x] - [(1-x^{-1})/(1-y^{-1})] + [(1-x)/(1-y)]$. The homomorphism $\psi : Q(F) \to F^* \otimes F^*$, $[x] \mapsto x \otimes (x-1)$ induces a homomorphism $\mathfrak{p}(F) \to (F^* \otimes F^*)_\sigma$, [13, 1.1]. By [13, 2.2], $E^2_{4,0}(2) \simeq \mathfrak{p}(F)$. It is not difficult to see that the

$E^2_{p,q}(2)$-terms have the following form

$$\begin{array}{cccccc} * & * & & & & \\ 0 & E^2_{1,2}(2) & * & & & \\ 0 & 0 & 0 & 0 & * & \\ 0 & 0 & 0 & 0 & \mathfrak{p}(F) & * \end{array}.$$

An easy calculation shows that $E^2_{1,2}(2) \subseteq H_2(F^*) \oplus (F^* \otimes F^*)_\sigma$. By [13, 2.4] $d^3_{4,0}(2) : E^3_{4,0}(2) \to E^3_{1,2}(2) \simeq E^2_{1,2}(2)$ is defined by $d^3_{4,0}(2)([x]) = x \wedge (x-1) - x \otimes (x-1)$. Because the spectral sequence converges to zero we see that $d^3_{4,0}(2)$ is surjective and so $E^2_{1,2}(2)$ is generated by elements of the form $x \wedge (x-1) - x \otimes (x-1) \in H_2(F^*) \oplus (F^* \otimes F^*)_\sigma$.

5. Homology of GL_4

Lemma 5.1. *$\tilde{E}^2_{p,0}(4)$ is trivial for $0 \le p \le 6$.*

Remark 2. Lemma 5.1 gives a positive answer to a question asked by Dupont for $n = 3$ (see [10, 4.12]).

Lemma 5.2. *$\tilde{E}^2_{p,1}(4)$ is trivial for $0 \le p \le 5$.*

Lemma 5.3. *$\tilde{E}^2_{p,2}(4)$ is trivial for $0 \le p \le 4$.*

Lemma 5.4. *$\tilde{E}^2_{p,3}(4)$ is trivial for $0 \le p \le 3$.*

The proof of these lemmas is lengthy calculation. We prove them in the next section.

Theorem 5.5. *The complex*

$$H_4({F^*}^2 \times \mathrm{GL}_2) \xrightarrow{\beta^{(4)}_2} H_4(F^* \times \mathrm{GL}_3) \xrightarrow{\beta^{(4)}_1} H_4(\mathrm{GL}_4) \to 0$$

is exact.

Proof. This follows from 5.1, 5.2, 5.3, 5.4 and the fact that the spectral sequence converges to zero. □

Theorem 5.6. (i) *If $\mathrm{char}(k) \neq 2, 3$, then $H_4(\mathrm{GL}_3, k) \xrightarrow{H_4(\mathrm{inc})} H_4(\mathrm{GL}_4, k)$ is injective.*

(ii) *If F be algebraically closed, then $H_4(\mathrm{GL}_3) \xrightarrow{H_4(\mathrm{inc})} H_4(\mathrm{GL}_4)$ is injective.*

Proof. (i) This follows from 5.5, [8, Thm. 4.2] and 4.1.
(ii) Since F is algebraically closed, $H_2(F)$ and $K^M_2(F)$ are uniquely divisible, so $\mathrm{Tor}^{\mathbb{Z}}_1(F^*, H_2(F^*)) = \mathrm{Tor}^{\mathbb{Z}}_1(F^*, K^M_2(F)) = 0$. Also from $H_2(\mathrm{GL}_3) \simeq H_2(\mathrm{GL}_2) = H_2(F^*) \oplus K^M_2(F)$ one sees that $\mathrm{Tor}^{\mathbb{Z}}_1(F^*, H_2(\mathrm{GL}_3)) = 0$. Therefore by the Künneth theorem

$$H_4(F^* \times \mathrm{GL}_3) \simeq \bigoplus_{i=0}^4 H_i(F^*) \otimes H_{4-i}(\mathrm{GL}_3),$$
$$H_4({F^*}^2 \times \mathrm{GL}_2) \simeq \bigoplus_{0 \le i+j \le 4} H_i(F^*_1) \otimes H_j(F^*_2) \otimes H_{4-i-j}(\mathrm{GL}_2).$$

Now the proof of the claim is similar to the proof of 4.1, using the fact that $H_2(\mathrm{inc}) : H_2(\mathrm{GL}_1) \to H_2(\mathrm{GL}_2)$, $H_3(\mathrm{inc}) : H_3(\mathrm{GL}_2) \to H_3(\mathrm{GL}_3)$ are injective [8, Thm. 4.2]. The only place where we need a modification is the definition of the map f. First we define

$$\nu'_3 : K_3^M(F) \to H_3(\mathrm{GL}_3), \quad \{a,b,c\} \mapsto [a^{1/3}, b, c].$$

This map is well defined as $K_3^M(F)$ is uniquely divisible. Set $f := \mathrm{id} \otimes \nu'_3 : H_1(F^*) \otimes K_3^M(F) \to H_1(F^*) \otimes H_3(\mathrm{GL}_3)$. The rest of the proof can be done similar to the proof of 4.1. We leave the detail to the reader. $\square$

Example 1. $H_4(\mathrm{inc}) : H_4(\mathrm{GL}_3(\mathbb{R}), \mathbb{Z}[\frac{1}{2}]) \to H_4(\mathrm{GL}_4(\mathbb{R}), \mathbb{Z}[\frac{1}{2}])$ is injective. It is well-known that $K_3^M(\mathbb{R}) = \langle \{-1,-1,-1\} \rangle \oplus V$, where $\langle \{-1,-1,-1\} \rangle$ is a group of order 2 generated by $\{-1,-1,-1\}$ and V is a uniquely divisible group. So the proof of our claim is similar to the proof of Theorem 5.6. (We invert 2 in the coefficient ring in order to eliminate the 2-torsion elements that appear in the decomposition of $H_4(F^* \times \mathrm{GL}_3)$ and $H_4(F^{*2} \times \mathrm{GL}_2)$. This might not be necessary.)

Corollary 5.7. (i) *If* $\mathrm{char}(k) \neq 2, 3$*, then*

$$0 \to H_4(\mathrm{GL}_3, k) \to H_4(\mathrm{GL}_4, k) \to K_4^M(F) \otimes k \to 0.$$

is split exact.

(ii) *If F is algebraicly closed, then*

$$0 \to H_4(\mathrm{GL}_3) \to H_4(\mathrm{GL}_4) \to K_4^M(F) \to 0.$$

is split exact.

Proof. The part (i) follows from 5.6, Suslin's exact sequence (see the introduction). For part (ii) we also need to know that $K_4^M(F)$ is uniquely divisible. It is easy to see that a splitting map can be defined by

$$\{a,b,c,d\} \mapsto -\frac{1}{6}[a,b,c,d], \quad \{a,b,c,d\} \mapsto [a^{-1/6}, b, c, d],$$

respectively. $\square$

Proposition 5.8. (i) *If* $\mathrm{char}(k) \neq 2, 3$*, then*

$$0 \to H_4(\mathrm{SL}_3, k)_{F^*} \to H_4(\mathrm{SL}_4, k)_{F^*} \to K_4^M(F) \otimes k \to 0.$$

is split exact.

(ii) *If F is algebraically closed, then*

$$0 \to H_4(\mathrm{SL}_3) \to H_4(\mathrm{SL}_4) \to K_4^M(F) \to 0.$$

is split exact.

Proof. The first part follows from 5.7, 2.3 and 2.4. For the proof of (ii), as in the proof of 2.4, it is sufficient to prove that the canonical map $H_4(\mathrm{SL}_3) \to H_4(\mathrm{GL}_3)$ is injective. For this we look at the Lyndon-Hochschild-Serre spectral sequence

$$E_{p,q}^2 = H_p(\mathrm{GL}_3, H_q(\mu_{3,F})) \Rightarrow H_{p+q}(F^* \times \mathrm{SL}_3)$$

obtained from the extension $1 \to \mu_{3,F} \to F^* \times \mathrm{SL}_3 \xrightarrow{\gamma_3} \mathrm{GL}_3 \to 0$, where $\gamma_3(a, A) := aA$. It is well-known that $H_i(F^*)$, $K_i^M(F)$ and $K_i(F)$ are divisible [5, Prop. 4.7], [2, 1.2], [11]. The Hurewicz map $K_i(F) \to H_i(\mathrm{SL})$ is isomorphism for $i = 2, 3$ (see [13, 5.2] or [10, 2. 5]). Since $B\mathrm{GL}^+ \sim BF^* \times B\mathrm{SL}^+$ [13, 5.3], one can see that $H_i(\mathrm{GL}_3)$ is divisible for $1 \le i \le 3$. Therefore the E^2-terms of the spectral sequence are of the following form

$$\begin{array}{ccccc}
\mathbb{Z}/3\mathbb{Z} & 0 & & & \\
0 & 0 & 0 & & \\
\mathbb{Z}/3\mathbb{Z} & 0 & \mathbb{Z}/3\mathbb{Z} & * & \\
0 & 0 & 0 & 0 & \\
\mathbb{Z}/3\mathbb{Z} & 0 & \mathbb{Z}/3\mathbb{Z} & 0 & * \\
\mathbb{Z} & H_1(\mathrm{GL}_3) & H_2(\mathrm{GL}_3) & H_3(\mathrm{GL}_3) & H_4(\mathrm{GL}_3).
\end{array}$$

An easy analysis of this spectral sequence shows that

$$H_3(\gamma_3) : H_3(F^* \times \mathrm{SL}_3) \to H_3(\mathrm{GL}_3)$$

is surjective with the kernel of order dividing 9. By the Künneth theorem

$$H_3(F^* \times \mathrm{SL}_3) \simeq H_3(\mathrm{SL}_3) \oplus F^* \otimes H_2(\mathrm{SL}_3) \oplus H_3(F^*).$$

Let $\xi \in F^*$ be the third root of unity, that is $\xi^3 = 1$ and $\xi \neq 1$. If we use the bar resolution $C_*(G)$ to define the homology of a group G (see [4, p. 36]), one can see that $\chi(\xi) := [\xi|\xi|\xi] + [\xi|\xi^2|\xi] \in H_3(F^*)$ has order 3. In fact in $C_*(G)_G$ $(G = F^*)$

$$\partial_4([\xi|\xi|\xi|\xi] + [\xi|\xi^2|\xi|\xi] + [\xi|\xi|\xi|\xi^2] + [\xi^2|\xi|\xi^2|\xi]) = 3\chi(\xi).$$

In a similar way we can define $\chi(\xi I_3) \in H_3(\mathrm{SL}_3)$ and $\chi(\xi I_3) \in H_3(\mathrm{GL}_3)$. Now it is easy to see that $H_3(\gamma_3)(\chi(\xi I_3), 0, 2\chi(\xi)) = 0$. Therefore the kernel of $H_3(\gamma_3)$ is not trivial. Thus $d_{4,0}^2$ or $d_{4,0}^4$ is trivial. In either case this implies that

$$H_4(F^* \times \mathrm{SL}_3) \to H_4(\mathrm{GL}_3).$$

is injective. Therefore $H_4(\mathrm{SL}_3) \to H_4(\mathrm{GL}_3)$ is injective.

To give a splitting map one should note that in $H_4(\mathrm{GL}_4)$, $[a, b, c, d]$ is equal to

$$\mathbf{c}(\mathrm{diag}(a, 1, a^{-1}, 1), \mathrm{diag}(b, b^{-1}, 1, 1), \mathrm{diag}(c, 1, c^{-1}, 1), \mathrm{diag}(d, d, d, d^{-3})).$$

We name this new version of $[a, b, c, d]$ by $[[a, b, c, d]]$. Then $[[a, b, c, d]] \in H_4(\mathrm{SL}_4)$ and a splitting map can be defined by

$$\{a, b, c, d\} \mapsto -\frac{1}{6}[[a, b, c, d]], \quad \{a, b, c, d\} \mapsto [[a^{-1/6}, b, c, d]],$$

respectively. □

Corollary 5.9. (i) *If* $\mathrm{char}(k) \neq 2, 3$, *then* $K_4(F)^{\mathrm{ind}} \otimes k$ *embeds in the group* $H_4(\mathrm{SL}_3, k)_{F^*}$.

(ii) *If* F *is algebraically closed, then* $K_4(F)^{\mathrm{ind}}$ *embeds in* $H_4(\mathrm{SL}_3)$.

Proof. (i) This follows from 5.6 and 2.4. Part (ii) can be proven in a similar way using the facts that the Hurewicz map $K_4(F) \to H_4(\mathrm{SL}_4)$ is injective [1, Thm. 7.23] and

$$0 \to K_4^M(F) \to K_4(F) \to K_4(F)^{\mathrm{ind}} \to 0$$

is split exact, for $K_4^M(F)$ [2, 1.2] and $K_4(F)$ [11] are uniquely divisible. □

6. Proof of lemmas 5.1, 5.2, 5.3, 5.4

In this section we will assume that $n = 4$. For simplicity we will write $\tilde{E}^1_{p,q}$ in place of $\tilde{E}^1_{p,q}(4)$ and so on. By $H_q(v)$ we mean $H_q(\mathrm{Stab}_{\mathrm{GL}_4}(v))$. Here we look at $F^* \simeq H_1(F^*)$ as an abelian group with multiplicative structure or as an abelian group with additive structure, depending on the context. This shall not cause any confusion. Note that under the isomorphism $F^* \to H_1(F^*)$ we have $1 \mapsto 0$ and $a^{-1} \mapsto -a$.

To prove the lemmas 5.1, 5.2, 5.3 and 5.4 we need to describe $\tilde{E}^1_{p,q}$ for $p = 3, 4, 5$. Let $w_i \in D_2(F^4)$, $u_i \in D_3(F^4)$ and $v_i \in D_4(F^4)$, where

$$w_1 = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle), \quad w_2 = (\langle e_1\rangle, \langle e_2\rangle, \langle e_1+e_2\rangle),$$

$$\begin{aligned}
&u_1 = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle), && u_2 = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_1+e_2+e_3\rangle),\\
&u_3 = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_1+e_2\rangle), && u_4 = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_2+e_3\rangle),\\
&u_5 = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_1+e_3\rangle), && u_6 = (\langle e_1\rangle, \langle e_2\rangle, \langle e_1+e_2\rangle, \langle e_3\rangle),\\
&u_{7,a} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_1+e_2\rangle, \langle e_1+ae_2\rangle), \quad a \in F^* - \{1\}, &&
\end{aligned}$$

$$\begin{array}{ll}
v_1 = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1+e_2+e_3+e_4\rangle) & \\
v_2^{i,j,k} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_i+e_j+e_k\rangle), & 1 \le i < j < k \le 4,\\
v_3^{i,j} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_i+e_j\rangle), & 1 \le i < j \le 4,\\
v_4 = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_1+e_2+e_3\rangle, \langle e_4\rangle), & \\
v_{5,a,b} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_1+e_2+e_3\rangle, & a, b \in F^*, a \ne 1\\
\qquad\qquad \langle e_1+ae_2+be_3\rangle), & \text{or } b \ne 1\\
v_{6,a}^{i,j} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_1+e_2+e_3\rangle, \langle e_i+ae_j\rangle), & 1 \le i < j \le 3, a \in F^*\\
v_7^{i,j} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_i+e_j\rangle, \langle e_4\rangle), & 1 \le i < j \le 3,\\
v_{8,a}^{i,j} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_i+ae_j\rangle, \langle e_1+e_2+e_3\rangle), & 1 \le i < j \le 3, a \in F^*\\
v_{9,a}^{i,j} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_i+e_j\rangle, \langle e_i+ae_j\rangle), & 1 \le i < j \le 3, a \in F^*\\
v_{10} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_1+e_2\rangle, \langle e_1+e_3\rangle), & \\
v_{11} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_1+e_3\rangle, \langle e_1+e_2\rangle), & \\
v_{12} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_1+e_2\rangle, \langle e_2+e_3\rangle), & \\
v_{13} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_2+e_3\rangle, \langle e_1+e_2\rangle), & \\
v_{14} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_1+e_3\rangle, \langle e_2+e_3\rangle), & \\
v_{15} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_2+e_3\rangle, \langle e_1+e_3\rangle), & \\
v_{16} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_1+e_2\rangle, \langle e_3\rangle, \langle e_4\rangle), &
\end{array}$$

$$\begin{array}{ll}
v_{17,a} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_1+e_2\rangle, \langle e_3\rangle, \langle e_1+ae_2+e_3\rangle), & a \in F^* \\
v_{18,a} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_1+e_2\rangle, \langle e_3\rangle, \langle e_1+ae_2\rangle), & a \in F^*-\{1\}, \\
v_{19} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_1+e_2\rangle, \langle e_3\rangle, \langle e_1+e_3\rangle), & \\
v_{20} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_1+e_2\rangle, \langle e_3\rangle, \langle e_2+e_3\rangle), & \\
v_{21,a} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_1+e_2\rangle, \langle e_1+ae_2\rangle, \langle e_3\rangle), & a \in F^*-\{1\}, \\
v_{22,a,b} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_1+e_2\rangle, \langle e_1+ae_2\rangle, & a,b \in F^*-\{1\}, a\neq b. \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \langle e_1+be_2\rangle), &
\end{array}$$

By the Shapiro lemma

$$\begin{array}{l}
\tilde{E}^1_{3,q} = H_q(w_1)\oplus H_q(w_2) \\
\tilde{E}^1_{4,q} = H_q(u_1)\oplus\cdots\oplus H_q(u_{7,a}) \\
\tilde{E}^1_{5,q} = H_q(v_1)\oplus\cdots\oplus H_q(v_{22,a,b}).
\end{array}$$

So by the center killer lemma [12, Thm. 1.9], we may assume that

$$\tilde{E}^1_{3,q} = H_q({F^*}^3\times \mathrm{GL}_1)\oplus H_q(F^*I_2\times \mathrm{GL}_2)$$

and so on. Note that

$$\begin{array}{ll}
\tilde{d}^1_{p,q} = d^1_{p,q} \text{ for } p=1,2, & \tilde{d}^1_{3,q}|_{H_q(w_1)} = d^1_{3,q}, \\
\tilde{d}^1_{3,q}|_{H_q(w_2)} = H_q(\mathrm{inc}), & \tilde{d}^1_{4,q}|_{H_q(u_1))} = (d^1_{4,q},0), \\
\tilde{d}^1_{4,q}|_{H_q(u_2)} = 0, & \tilde{d}^1_{4,q}|_{H_q(u_3)} = (-H_q(\mathrm{inc}), H_q(\mathrm{inc})), \\
\tilde{d}^1_{4,q}|_{H_q(u_4)} = (-H_q(\mathrm{inc}), H_q(\alpha)), & \tilde{d}^1_{4,q}|_{H_q(u_5)} = (-H_q(\beta), H_q(\gamma)), \\
\tilde{d}^1_{4,q}|_{H_q(u_6)} = (H_q(\mathrm{inc}), H_q(\mathrm{inc})), & \tilde{d}^1_{4,q}|_{H_q(u_{7,a})} = 0,
\end{array}$$

where

$$\begin{array}{l}
\alpha : (a,b,b,c)\mapsto (b,b,a,c), \\
\beta : (a,b,a,c)\mapsto (b,a,a,c), \\
\gamma : (a,b,a,c)\mapsto (a,a,b,c).
\end{array}$$

Lemma 5.1 *The group $\tilde{E}^2_{p,0}$ is trivial for $0\le p\le 6$.*

Proof. The triviality of $\tilde{E}^2_{6,0}$ is the most difficult one, which we prove it here. The rest is much easier and we leave it to the readers.
Triviality of $\tilde{E}^2_{6,0}$. The proof is in four steps;
Step 1. The sequence $0\to C_*(F^4)_{\mathrm{GL}_4}\to D_*(F^4)_{\mathrm{GL}_4}\to Q_*(F^4)\mathrm{GL}_4\to 0$ is exact, where $Q_*(F^4) := D_*(F^4)/C_*(F^4)$.
Step 2. The group $H_5(Q_*(F^4)_{\mathrm{GL}_4})$ is trivial.
Step 3. The map induced in homology by $C_*(F^4)_{\mathrm{GL}_4}\to D_*(F^4)_{\mathrm{GL}_4}$ is zero in degree 5.
Step 4. The group $\tilde{E}^2_{6,0}$ is trivial.
Proof of step 1. For $i\ge -1$, $D_i(F^4)\simeq C_i(F^4)\oplus Q_i(F^4)$. This decomposition is compatible with the action of GL_4, so we get an exact sequence of $\mathbb{Z}[\mathrm{GL}_4]$-modules

$$0\to C_i(F^4)\to D_i(F^4)\to Q_i(F^4)\to 0$$

which splits as a sequence of $\mathbb{Z}[\mathrm{GL}_4]$-modules. One can easily deduce the desired exact sequence from this. Note that this exact sequence does not split as complexes.

Proof of step 2. The complex $Q_*(F^4)$ induces a spectral sequence

$$\hat{E}^1_{p,q} = \begin{cases} 0 & \text{if } 0 \leq p \leq 2 \\ H_q(\mathrm{GL}_4, Q_{p-1}(F^4)) & \text{if } p \geq 3 \end{cases}$$

which converges to zero. To prove the claim it is sufficient to prove that $\hat{E}^2_{6,0} = 0$ and to prove this it is sufficient to prove that $\hat{E}^2_{4,1} = \hat{E}^2_{3,2} = 0$. One can see that

$$\begin{aligned} \hat{E}^1_{3,q} &= H_q(w_2) \\ \hat{E}^1_{4,q} &= H_q(u_2) \oplus \cdots \oplus H_q(u_{7,a}), \\ \hat{E}^1_{5,q} &= H_q(v_2^{i,j,k}) \oplus \cdots \oplus H_q(v_{22,a,b}). \end{aligned}$$

It is easy to see that $\hat{d}^1_{4,2}|_{H_2(u_6)} = H_2(\mathrm{inc})$, so it is surjective. Thus $\hat{E}^2_{3,2} = 0$. Easy calculation shows that, $d \in F^* - \{1\}$ fixed,

$$\hat{d}^1_{5,1} : H_1(v_{18,d}) \to \hat{E}^1_{4,1}, \ \ y \mapsto (0, y, 0, 0, y, -y), \tag{1}$$

$$\hat{d}^1_{5,1} : H_1(v_3^{1,3}) \to \hat{E}^1_{4,1}, \quad y \mapsto (0, *, 0, y, 0, y), \tag{2}$$

$$\hat{d}^1_{5,1} : H_1(v_{9,a}^{1,2}) \to \hat{E}^1_{4,1}, \quad y \mapsto (0, 0, 0, 0, 0, y), \tag{3}$$

$$\hat{d}^1_{5,1} : H_1(v_3^{2,3}) \to \hat{E}^1_{4,1}, \quad (a, b, b, c) \mapsto (0, (b, b, c, a), *, 0, 0, 0), \tag{4}$$

$$\hat{d}^1_{5,1} : H_1(v_4) \to \hat{E}^1_{4,1}, \qquad y \mapsto (y, 0, 0, 0, 0, 0), \tag{5}$$

$$\hat{d}^1_{5,1} : H_1(v_3^{3,4}) \to \hat{E}^1_{4,1}, \quad (0, a, 0, 0) \mapsto (0, (a, 0, 0, -a), 0, 0, 0, 0). \tag{6}$$

Let $x = (x_2, x_3, \dots, x_{7,a}) \in \ker(\hat{d}^1_{4,1})$. By (1), (2), (3), (4) and (5) we may assume $x_6 = 0$, $x_5 = 0$, $x_{7,a} = 0$, $x_3 = 0$, $x_2 = 0$, respectively. If $x_4 = (a, b, b, c)$, then $\hat{d}^1_{4,1}(x_4) = (b, b, \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}) = 0$. Hence $b = 0$ and $c = -a$. Now by (6), $\hat{d}^1_{5,1}((0, a, 0, 0)) = (0, 0, x_4, 0, 0, 0) = x$. Therefore $\hat{E}^2_{4,1} = 0$.

Proof of step 3. Here all the calculation will take place in $C_*(F^4)_{\mathrm{GL}_4}$ and $D_*(F^4)_{\mathrm{GL}_4}$. For simplicity the image of $v \in C_*(F^4)$ in $C_*(F^4)_{\mathrm{GL}_4}$ is denoted by v. Consider the following commutative diagram

$$\begin{array}{ccccc} C_6(F^4)_{\mathrm{GL}_4} & \to & C_5(F^4)_{\mathrm{GL}_4} & \to & C_4(F^4)_{\mathrm{GL}_4} \\ \downarrow & & \downarrow & & \downarrow \\ D_6(F^4)_{\mathrm{GL}_4} & \to & D_5(F^4)_{\mathrm{GL}_4} & \to & D_4(F^4)_{\mathrm{GL}_4}. \end{array}$$

The generators of $C_5(F^4)_{\mathrm{GL}_4}$ are of the form $x_{a,b,c}$ with

$$x_{a,b,c} = (\langle e_1 \rangle, \langle e_2 \rangle, \langle e_3 \rangle, \langle e_4 \rangle, \langle e_1 + e_2 + e_3 + e_4 \rangle, \langle e_1 + ae_2 + be_3 + ce_4 \rangle),$$

where $a,b,c \in F^* - \{1\}$ and $a \neq b$, $a \neq c$, $b \neq c$. Since $C_4(F^4) \otimes_{\mathrm{GL}_4} \mathbb{Z} = \mathbb{Z}$, $x_{a,b,c} \in \ker(\partial_5)$ and the elements of this form generate $\ker(\partial_5)$. Hence to prove this step it is sufficient to prove that $x_{a,b,c} \in \mathrm{im}(\tilde{\partial}_6)$. Let

$$X_{a,b,c} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1{+}e_2{+}e_3{+}e_4\rangle, \langle e_1{+}ae_2{+}be_3{+}ce_4\rangle, \langle e_1{+}e_2\rangle),$$

where a,b,c are as above. Then

$$\tilde{\partial}_6(X_{a,b,c}) = v_{\frac{1-b}{1-c},1-b} - v_{\frac{a-b}{a-c},a-b} + u_{\frac{1-b}{a-b}} - u_{\frac{1-c}{a-c}} + u_{\frac{1}{a}} + x_{a,b,c},$$

where

$$\begin{aligned} v_{g,h} &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1+e_2+e_3+e_4\rangle, \langle e_2+ge_3+he_4\rangle),\\ u_l &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1+e_2+e_3+e_4\rangle, \langle e_1+le_2\rangle)\\ &\quad -(\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1+e_2+e_3+e_4\rangle, \langle e_1+e_2\rangle), \end{aligned}$$

$g,h,l \in F^* - \{1\}, g \neq h$. So it is sufficient to prove that $v_{g,h} - v_{p,q}$ and u_l are in the image if $\tilde{\partial}_6$. Let

$$\begin{aligned} U_l &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1+e_2+e_3+e_4\rangle, \langle e_1+e_2\rangle, \langle e_1+le_2\rangle),\\ V_l &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1+e_2\rangle, \langle e_1+le_2\rangle, \langle e_3+e_4\rangle), \end{aligned}$$

where $l \in F^* - \{1\}$. Then $\tilde{\partial}_6((V_l - U_l)) = u_l$. Set

$$T_{g,h} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1+e_2+e_3+e_4\rangle, \langle e_2+ge_3+he_4\rangle, \langle e_2+e_3\rangle),$$

where $g,h \in F^* - \{1\}, g \neq h$. Then

$$\begin{aligned} \tilde{\partial}_6(T_{g,h} - T_{p,q}) = {}& -s_{\frac{1}{1-h}} + s_{\frac{g}{g-h}} + s_{\frac{1}{1-q}} - s_{\frac{p}{p-q}} - z_{\frac{1-h}{g-h}} + z_{\frac{1-q}{p-q}}\\ & + y_{\frac{1}{g}} - y_{\frac{1}{p}} + v_{g,h} - v_{p,q}, \end{aligned}$$

where

$$\begin{aligned} s_a &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1+e_2+e_3+e_4\rangle, \langle e_1+ae_3+e_4\rangle),\\ z_a &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1+e_2+e_3+e_4\rangle, \langle e_2+ae_3\rangle)\\ &\quad -(\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1+e_2+e_3+e_4\rangle, \langle e_2+e_3\rangle),\\ y_a &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1+e_2+e_3\rangle, \langle e_1+ae_2\rangle)\\ &\quad +(\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_2+e_3+e_4\rangle, \langle e_2+ae_3\rangle)\\ &\quad -(\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1+e_2+e_3\rangle, \langle e_1+e_2\rangle)\\ &\quad -(\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_2+e_3+e_4\rangle, \langle e_2+e_3\rangle). \end{aligned}$$

So to prove that $v_{g,h} - v_{p,q} \in \mathrm{im}(\tilde{\partial}_6)$ it is sufficient to prove that $s_a - s_b, a \neq b$, $z_a, y_a \in \mathrm{im}(\tilde{\partial}_6)$, $a,b \in F^* - \{1\}$. Set

$$\begin{aligned} Y_a &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1+e_2+e_3\rangle, \langle e_1+ae_2\rangle, \langle e_1+e_2\rangle),\\ Y'_a &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_2+e_3+e_4\rangle, \langle e_2+ae_3\rangle, \langle e_2+e_3\rangle). \end{aligned}$$

Then $y_a = \tilde{\partial}_6(Y_a + Y'_a) - 2\tilde{\partial}_6(V_{\frac{1}{a}})$. To prove that $z_a \in \mathrm{im}(\tilde{\partial}_6)$, set

$$Z_a = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1+e_2+e_3+e_4\rangle, \langle e_2+e_3\rangle, \langle e_2+ae_3\rangle).$$

By an easy calculation $z_a = \tilde{\partial}_6(V_a) - \tilde{\partial}_6(Z_a)$. If

$$S_a = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1+e_2+e_3+e_4\rangle, \langle e_1+ae_3+e_4\rangle, \langle e_1+e_4\rangle),$$

then $\tilde{\partial}_6(S_a - S_b) = R_{\frac{1}{1-a},\frac{1}{1-b}} + s_a - s_b$, where, $a, b \in F^* - \{1\}, a \neq b$ and

$$\begin{aligned} R_{a,b} = (&\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1+e_2+e_4\rangle, \langle e_1+ae_2+e_4\rangle) \\ -(&\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1+e_2+e_4\rangle, \langle e_1+be_2+e_4\rangle) \\ -(&\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_2+e_3+e_4\rangle, \langle e_2+ae_3+e_4\rangle) \\ +(&\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_2+e_3+e_4\rangle, \langle e_2+be_3+e_4\rangle). \end{aligned}$$

Thus to prove that $s_a - s_b \in \mathrm{im}(\tilde{\partial}_6)$ it is sufficient to prove that $R_{a,b} \in \mathrm{im}(\tilde{\partial}_6)$. Let

$$\begin{aligned} Q_a &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1+e_2+e_4\rangle, \langle e_1+ae_2+e_4\rangle, \langle e_1+e_4\rangle) \\ &\quad -(\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_2+e_3+e_4\rangle, \langle e_2+ae_3+e_4\rangle, \langle e_1+e_4\rangle), \\ N_{a,b} &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1+e_4\rangle, \langle e_1+ae_4\rangle) \\ &\quad -(\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1+e_4\rangle, \langle e_1+be_4\rangle), \\ P_a &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_3+e_4\rangle, \langle ae_3+e_4\rangle). \end{aligned}$$

Then $\tilde{\partial}_6(Q_a - Q_b) = N_{\frac{1}{1-a},\frac{1}{1-b}} + P_{\frac{1}{1-a}} + P_{\frac{1}{1-b}} + R_{a,b}$. If

$$\begin{aligned} O_{a,b} &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1+e_4\rangle, \langle e_1+ae_4\rangle, \langle e_2+e_3\rangle) \\ &\quad -(\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1+e_4\rangle, \langle e_1+be_4\rangle, \langle e_2+e_3\rangle), \\ M_a &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_3+e_4\rangle, \langle ae_3+e_4\rangle, \langle e_1+e_2\rangle), \end{aligned}$$

then $\tilde{\partial}_6(O_{a,b}) = N_{a,b}$ and $\tilde{\partial}_6(M_a) = P_a$. This completes the proof of step 3.
Proof of Step 4. writing the homological long exact sequence of the short exact sequence obtained in the first step, we get the exact sequence

$$H_5(C_*(F^4)_{\mathrm{GL}_4}) \to H_5(D_*(F^4)_{\mathrm{GL}_4}) \to H_5(Q_*(F^4)_{\mathrm{GL}_4}).$$

By steps 2 and 3, $H_5(D_*(F^4)_{\mathrm{GL}_4}) = 0$, but $\tilde{E}^2_{6,0} = H_5(D_*(F^4)_{\mathrm{GL}_4})$. This completes the proof of the triviality of $\tilde{E}^2_{6,0}$. □

Lemma 5.2 $\tilde{E}^2_{p,1}(4)$ *is trivial for* $0 \leq p \leq 5$.

Proof. The case $p = 5$ is the most difficult case and we show that $\tilde{E}^2_{5,1}(4) = 0$. We leave the rest to the reader. Let $x = (x_1, \dots, x_{22,a,b}) \in \ker(\tilde{d}^1_{5,1})$. Set

$$V_{22,a,b} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_1+e_2\rangle, \langle e_1+ae_2\rangle, \langle e_1+be_2\rangle),$$

where $a, b \in F^* - \{1\}, a \neq b$. Then

$$\tilde{d}^1_{6,1} : H_1(V_{22,a,b}) \to H_1(v^{1,2}_{9,a'}) \oplus H_1(v_{22,a,b}), \quad y \mapsto (*, y).$$

So $x - \tilde{d}^1_{6,1}(x_{22,a,b}) = (x'_1, \dots, x'_{21,a}, 0)$. So we may assume $x_{22,a,b} = 0$. In a similar way we may assume $x_{5,a,b} = x^{i,j}_{6,a} = x^{i,j}_{8,a} = x^{i,j}_{9,a} = x_{10} = x_{11} = x_{12} = x_{13} = x_{14} = x_{15} = x_{19} = x_{17,a} = x_{20} = x^{1,2}_7 = x_{16} = 0$. Choose $a, b \in F^* - \{1\}$, $a \neq b$ and set

$$\begin{aligned} {}_1V^{1,3}_7 &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1+e_4\rangle, \langle e_2+ae_4\rangle), \\ {}_2V^{1,3}_7 &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_1+e_3\rangle, \langle e_4\rangle, \langle e_1+ae_3\rangle), \\ {}_2V^{1,3}_7 &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_1+e_2\rangle, \langle e_1+ae_2\rangle, \langle e_3\rangle, \langle e_1+be_2\rangle), \\ {}_3V^{1,3}_7 &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_2+e_3\rangle, \langle e_2+ae_3\rangle, \langle e_2+be_3\rangle). \end{aligned}$$

Then

$$\begin{array}{l}\tilde{d}^1_{6,1}: \bigoplus_{i=1}^4 {}_iV_7^{1,3} \to H_1(v_3^{i,j}) \oplus H_1(v_7^{1,3}) \oplus H_1(v_{18,a}) \oplus H_1(v_{21,a}),\\ ((a,b,c,b), -(b,c,b,a), -(b,b,c,a), (a,b,b,c)) \mapsto (*, (b,c,b,a), *, *).\end{array}$$

So we may assume $x_7^{1,3} = 0$. If

$$\begin{aligned}{}_1V_2^{1,2,3} &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1+e_2+e_3\rangle), \langle e_1+e_2\rangle),\\ {}_2V_2^{1,2,3} &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1+e_2+e_4\rangle), \langle e_1+e_2\rangle),\end{aligned}$$

then

$$\begin{array}{l}\tilde{d}^1_{6,1}: \bigoplus_{i=1}^2 {}_iV_2^{1,2,3} \to H_1(v_3^{1,2}) \oplus H_1(v_2^{1,2,3}) \oplus H_1(v_2^{1,2,4}),\\ ((a,a,a,b), (a,a,b,a)) \mapsto ((2a, 2a, a+b, a+b), -(a,a,a,b), (a,a,b,a)).\end{array}$$

So we may assume $x_2^{1,2,3} = 0$. In a similar way we may assume $x_2^{2,3,4} = x_2^{1,3,4} = x_7^{1,2} = x_7^{1,3} = x_{16} = x_{18,a} = 0$. Therefore we reduce x to

$$x = (x_1, x_2^{1,2,4}, x_3^{i,j}, x_4, x_7^{2,3}, x_{21,a}) \in \ker(\tilde{d}^1_{5,1}).$$

Set

$$\begin{array}{lll} x_1 = (a'', a'', a'', a''), & x_2^{1,2,4} = (s,t,s,s), & x_3^{1,2} = (a_0, a_0, b_0, c_0),\\ x_3^{1,3} = (a_1, b_1, a_1, c_1), & x_3^{1,4} = (a_2, b_2, c_2, a_2), & x_3^{2,3} = (a_3, b_3, b_3, c_3),\\ x_3^{2,3} = (a_4, b_4, c_4, b_4), & x_3^{3,4} = (a_5, b_5, c_5, c_5), & x_4 = s', s', s', t'),\\ x_{21,a} = (a', a', b', c'). & & \end{array}$$

Since $\tilde{d}^1_{5,1}(x) = 0$, one gets $x_7^{2,3} = 0$, $a' = 0$, $c' = -b'$, $s = -s'$, $t = -t'$, $a_1 = b_3 = b_4$. Let $a \in F^* - \{1\}$ and fix $d \in F^* - \{1\}$. If

$$\begin{array}{l} V_{21,a} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_1+e_2\rangle, \langle e_1+ae_2\rangle, \langle e_3\rangle, \langle e_4\rangle),\\ V_4 = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_1+e_2+e_3\rangle, \langle e_4\rangle, \langle e_1+e_2+de_3\rangle),\\ V_2^{1,2,4} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1+e_2+e_3\rangle, \langle e_1+e_2+de_3\rangle),\end{array}$$

then

$$\begin{array}{l} \tilde{d}^1_{6,1}: H_1(V_{21,a}) \to H_1(v_{21,a}), \quad (0,0,-b',0) \mapsto, (0,0,b',-b'),\\ \tilde{d}^1_{6,1}: H_1(V_2^{1,2,4}) \oplus H_1(V_4) \to H_1(v_2^{1,2,4}) \oplus H_1(v_4),\\ \qquad ((s,s,s,t),(s,s,s,t)) \mapsto (-(s,s,t,s),(s,s,s,t)).\end{array}$$

So we may assume $x_4 = x_2^{1,2,4} = x_{21,a} = 0$. Let

$$V = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1+e_2+e_3+e_4\rangle, \langle e_2+be_4\rangle).$$

Then $\tilde{d}^1_{6,1}: H_1(V) \to H_1(v_1) \oplus H_1(v_3^{1,3}) \oplus H_1(v_3^{2,3}) \oplus H_1(v_3^{2,4})$, where $y = (a,a,a,a) \mapsto (y,y,y,y)$. In this way we may assume $a_1 = b_3 = b_4 = 0$. In a similar way we may assume $a_0 = c_5 = 0$. If

$$V_1 = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1+e_2+e_3+e_4\rangle, \langle e_1+be_4\rangle),$$

then $\tilde{d}^1_{6,1}: H_1(V_1) \to H_1(v_1) \oplus H_1(v_3^{1,4})$, $y = (a'', a'', a'', a'') \mapsto (-y, y)$. So we may assume $x_1 = 0$. Thus $x = (x_3^{1,2}, x_3^{1,3}, x_3^{1,4}, x_3^{2,3}, x_3^{2,4}, x_3^{3,4})$, where

$$\begin{aligned} x_3^{1,2} &= (0,0,b_0,c_0), \quad x_3^{1,3} = (0,b_1,0,c_1), \quad x_3^{1,4} = (a_2,b_2,a_2,c_2),\\ x_3^{2,3} &= (a_3,0,0,c_3), \quad x_3^{2,4} = (a_4,0,c_4,0), \quad x_3^{3,4} = (a_5,b_5,0,0). \end{aligned}$$

From $\tilde{d}^1_{5,1}(x) = 0$, we have the following relations

$$\begin{array}{ll} b_1 + b_2 + a_3 - a_4 + a_5 = 0, & 2b_1 + c_2 - c_4 + b_5 = 0,\\ b_0 + 2c_1 + a_2 = 0, & c_0 + c_1 + c_3 = 0,\\ c_0 - b_0 - c_1 + c_3 = 0, & b_0 - c_0 - b_1 + a_3 = 0,\\ -a_3 + a_4 - b_5 - a_5 = 0, & -c_3 + c_4 - a_5 - b_5 = 0,\\ -b_1 + b_2 - c_2 + c_4 = 0, & -c_1 + c_2 - b_2 + a_4 = 0. \end{array}$$

Set

$$H_1(v_3^{i,j}) = H_1(v_3^{1,2}) \oplus H_1(v_3^{1,3}) \oplus H_1(v_3^{1,4}) \oplus H_1(v_3^{2,3}) \oplus H_1(v_3^{2,4}) \oplus H_1(v_3^{3,4}).$$

Let

$$\begin{aligned} W_1 &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_1+e_2+e_3\rangle, \langle e_4\rangle, \langle e_2+e_4\rangle),\\ W_2 &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1+e_2+e_4\rangle, \langle e_2+e_4\rangle),\\ W_3 &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1+e_3+e_4\rangle, \langle e_1+e_4\rangle),\\ W_4 &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_2+e_3+e_4\rangle, \langle e_2+e_4\rangle),\\ W_5 &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1+e_3+e_4\rangle, \langle e_3+e_4\rangle),\\ W_6 &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_2+e_3+e_4\rangle, \langle e_3+e_4\rangle). \end{aligned}$$

Then $\tilde{d}^1_{6,1}: H_1(W_1) \oplus H_1(W_2) \oplus H_1(V_4) \oplus H_1(V_2^{1,2,4}) \to H_1(v_3^{i,j})$, where

$$\begin{aligned} &((0,0,0,c_3), (0,0,c_3,0), (0,0,0,c_3), (0,0,0,c_3)) \mapsto\\ &\qquad ((0,0,0,c_3), -(0,c_3,0,0), (0,c_3,c_3,0), -(0,0,0,c_3), -(0,0,c_3,0), 0) \end{aligned}$$

and $\bigoplus_{i=3}^6 H_1(W_i) \to H_1(v_3^{i,j})$, where

$$\begin{aligned} &((0,a_3,0,0), (a_3,0,0,0), -(0,a_3,0,0), -(a_3,0,0,0)) \mapsto\\ &\qquad (0, (0,a_3,0,0), (0,a_3,0,0), (a_3,0,0,0), (a_3,0,0,0), -2(a_3,a_3,0,0)). \end{aligned}$$

So we may assume $a_3 = c_3 = 0$,that is $x_3^{2,3} = 0$. From the above equations we have $a_4 = -c_4 = a_5 - b_5$. Let

$$\begin{aligned} {}_1V_3^{2,4} &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1+e_3\rangle, \langle e_1+ae_3\rangle),\\ {}_1V_3^{2,4} &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_2+e_4\rangle, \langle e_2+ae_4\rangle),\\ {}_1V_3^{2,4} &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_2+e_3\rangle, \langle e_2+ae_3\rangle),\\ {}_1V_3^{2,4} &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_3+e_4\rangle, \langle e_3+ae_4\rangle), \end{aligned}$$

Under the map $\tilde{d}^1_{6,1}: \bigoplus_{i=1}^4 H_1({}_iV_3^{2,4}) \to H_1(v_3^{i,j})$ we have

$$\begin{aligned} &((0,b,0,c), (c,0,b,0), (b,0,0,c), -(b,c,0,0)) \mapsto\\ &\qquad\qquad (0,0,(0,b,c,0),0,(b-c,0,c-b,0), -(c,b,0,0)). \end{aligned}$$

So we may assume $a_4 = c_4 = 0$, that is $x_3^{2,4} = 0$. If

$$V_3^{3,4} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1 + e_2\rangle, \langle e_3 + e_4\rangle),$$

then for $H_1(V_3^{3,4}) \to H_1(v_3^{i,j})$ we have

$$(a, a, 0, 0) \mapsto (-(a, a, 0, 0), (a, a, 0, 0), 0, 0, 0, 0).$$

So we may assume $a_5 = b_5 = 0$, that is $x_3^{3,4} = 0$. Therefore $b_1 = -c_1 = b_2 - c_2 = b_0 - c_0$. Let

$$\begin{aligned}
W_7 &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1 + e_2 + e_3\rangle, \langle e_1 + e_3\rangle),\\
W_8 &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1 + e_3 + e_4\rangle, \langle e_1 + e_4\rangle),\\
W_9 &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1 + e_2 + e_3\rangle, \langle e_1 + e_2\rangle),\\
W_{10} &= (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1 + e_3 + e_4\rangle, \langle e_1 + e_3\rangle).
\end{aligned}$$

Then under the map $\tilde{d}_{6,1}^1 : \bigoplus_{i=7}^{10} H_1(W_i) \to H_1(v_3^{i,j})$ we have

$$\begin{aligned}
z_1 := ((0, 0, 0, a), -(0, a, 0, 0), -(0, 0, 0, a), (0, a, 0, 0)) \mapsto&\\
(-(0, 0, 2a, a), (0, -a, 0, a), (0, -a, a, 0), 0, (-a, 0, a, 0), (a, 2a, 0, 0)).&
\end{aligned}$$

If $z_2 = ((0, 2a, 0, a), (a, 0, 2a, 0), (2a, 0, 0, a), -(2a, a, 0, 0)) \in \bigoplus_{i=1}^4 H_1({}_iV_3^{2,4})$, then $\tilde{d}_{6,1}^1(z_1 + z_2) = (-(0, 0, 2a, a), (0, -a, 0, a), (0, a, 2a, 0), 0, 0, 0)$. So we may assume $b_1 = c_1 = 0$. Thus $a_2 = c_2 = 0$ and $b_0 = c_0 = -a_2$. If $V_3^{1,2} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1 + e_4\rangle, \langle e_2 + e_3\rangle)$, then under $\tilde{d}_{6,1}^1 : H_1(V_3^{1,2}) \to H_1(v_3^{i,j})$ we have $(b_0, 0, 0, b_0) \mapsto ((0, 0, b_0, b_0), 0, -(b_0, 0, 0, b_0), 0, 0)$. This proves that $x \in \mathrm{im}(\tilde{d}_{6,1}^1)$, and therefore $\tilde{E}_{5,1}^2(4)$ is trivial. □

Lemma 5.3 *$\tilde{E}_{p,2}^2$ is trivial for $0 \leq p \leq 4$.*

Proof. The difficult case is $p = 4$. Let $x = (x_1, \dots, x_{7,a}) \in \ker(\tilde{d}_{4,2}^1)$. Consider the following maps

$$\begin{array}{ll}
\tilde{d}_{4,2}^1 : H_2(v_{18,a}) \to H_2(u_3) \oplus H_2(u_6) \oplus H_2(u_{7,a}), & y \mapsto (y, y, -y)\\
\tilde{d}_{4,2}^1 : H_2(v_{9,a}^{1,2}) \twoheadrightarrow H_2(u_{7,a}), & \\
\tilde{d}_{4,2}^1 : H_2(v_3^{1,3}) \to H_2(u_1) \oplus H_2(u_3) \oplus H_2(u_5), & y \mapsto (*, *, y),\\
\tilde{d}_{4,2}^1 : H_2(v_3^{2,3}) \to H_2(u_1) \oplus H_2(u_3) \oplus H_2(u_4), & y \mapsto (y, *, -y)\\
\tilde{d}_{4,2}^1 : H_2(v_4) \to H_2(u_1) \oplus H_2(u_2), & y \mapsto (0, y).
\end{array}$$

So we may assume $x_6 = x_{7,a} = x_5 = x_3 = x_2 = 0$. Hence $x = (x_1, x_4) \in H_2(u_1) \oplus H_2(u_4)$. Let $H_2(u_1) = H_2(F^{*3} \times \mathrm{GL}_1) = \bigoplus_{i=1}^{10} T_i$ and $H_2(u_4) = H_2(F^* I_2 \times F^* \times \mathrm{GL}_2) = \bigoplus_{i=11}^{16} T_i$, where

$$\begin{array}{llll}
T_1 = H_2(F_1^*), & T_2 = H_2(F_2^*), & T_3 = H_2(F_3^*), & T_4 = H_2(F_4^*),\\
T_5 = F_1^* \otimes F_2^*, & T_6 = F_1^* \otimes F_3^*, & T_7 = F_1^* \otimes F_4^*, & T_8 = F_2^* \otimes F_3^*,\\
T_9 = F_2^* \otimes F_4^*, & T_{10} = F_3^* \otimes F_4^*, & T_{11} = H_2(F^*), & T_{12} = H_2(F^* I_2),\\
T_{13} = H_2(\mathrm{GL}_1), & T_{14} = F^* \otimes F^* I_2, & T_{15} = F^* \otimes \mathrm{GL}_1, & T_{16} = F^* I_2 \otimes \mathrm{GL}_1.
\end{array}$$

Set $x=(x_1,x_4)=(z_1,\dots,z_{16})$ with $z_i\in T_i$. We look at the maps

$$\tilde{d}^1_{4,2}: H_2(v_3^{1,2})=H_2(F^*I_2\times F^*\times \mathrm{GL}_1)\rightarrow \bigoplus_{i=1}^{16} T_i,$$
$$(y_1,y_2,y_3,y_4,y_5,y_6)\mapsto (y_1,y_1,y_2,y_3,0,y_4,y_5,y_4,y_5,y_6,0,0,0,0,0,0).$$

and

$$\tilde{d}^1_{4,2}: H_2(v_3^{3,4})=H_2(F^*\times F^*\times F^*I_2)\rightarrow \bigoplus_{i=1}^{16} T_i,$$
$$(y,y',0)\mapsto (0,0,0,0,y,y',y',0,0,0,0,0,0,0,0,0).$$

where $H_2(v_3^{3,4})=H_2(F^*\times F^*\times F^*I_2)=F_1^*\otimes F_2^*\oplus F_1^*\otimes F^*I_2\oplus S$. So we may assume $z_2=z_3=z_4=z_8=z_9=z_{10}=0$, $z_5=z_6=0$. Now easy computation shows that

$$\begin{array}{ccc} -z_1-z_{11}=0, & -z_{12}=0, & -z_{12}+z_{13}=0,\\ z_1-z_{13}=0, & z_{14}=0. & \end{array}$$

Thus $z_1=z_{11}=z_{12}=z_{13}=z_{14}=0$. Now it is easy to see that the rest of z_i's should be zero. Thus $\tilde{E}^1_{4,2}=0$. □

Lemma 5.4 $\tilde{E}^2_{p,3}$ *is trivial for* $0\le p\le 3$.

Proof. Let $\tilde{E}^1_{2,3}=\bigoplus_{i=1}^{13}T_i$ and $\tilde{E}^1_{3,3}=\bigoplus_{i=1}^{26}S_i\oplus S'$ where

$$\begin{array}{lll} T_1=H_3(\mathrm{GL}_2), & T_2=H_1(F_1^*)\otimes H_2(\mathrm{GL}_2), & T_3=H_2(F_1^*)\otimes H_1(\mathrm{GL}_2),\\ T_4=H_3(F_1^*), & T_5=H_1(F_2^*)\otimes H_2(\mathrm{GL}_2), & T_6=H_2(F_2^*)\otimes H_1(\mathrm{GL}_2),\\ T_7=H_3(F_2^*), & T_8=H_1(F_1^*)\otimes H_2(F_2^*), & T_9=H_2(F_1^*)\otimes H_1(F_2^*), \end{array}$$

$$\begin{array}{ll} T_{10}=H_1(F_1^*)\otimes H_1(F_2^*)\otimes H_1(\mathrm{GL}_2), & T_{11}=\mathrm{Tor}_1^{\mathbb{Z}}(H_1(F_1^*),H_1(F_2^*)),\\ T_{12}=\mathrm{Tor}_1^{\mathbb{Z}}(H_1(F_1^*),H_1(\mathrm{GL}_2)), & T_{13}=\mathrm{Tor}_1^{\mathbb{Z}}(H_1(F_2^*),H_1(\mathrm{GL}_2)), \end{array}$$

and

$$\begin{array}{lll} S_1=H_3(\mathrm{GL}_1), & S_2=H_1(F_1^*)\otimes H_2(\mathrm{GL}_1), & S_3=H_2(F_1^*)\otimes H_1(\mathrm{GL}_1),\\ S_4=H_3(F_1^*), & S_5=H_1(F_2^*)\otimes H_2(\mathrm{GL}_1), & S_6=H_2(F_2^*)\otimes H_1(\mathrm{GL}_1),\\ S_7=H_3(F_2^*), & S_8=H_1(F_3^*)\otimes H_2(\mathrm{GL}_1), & S_9=H_2(F_3^*)\otimes H_1(\mathrm{GL}_1),\\ S_{10}=H_3(F_3^*), & S_{11}=H_1(F_1^*)\otimes H_2(F_2^*), & S_{12}=H_2(F_1^*)\otimes H_1(F_2^*), \end{array}$$

$$\begin{array}{ll} S_{13}=H_1(F_1^*)\otimes H_2(F_3^*), & S_{14}=H_2(F_1^*)\otimes H_1(F_3^*),\\ S_{15}=H_1(F_2^*)\otimes H_2(F_3^*), & S_{16}=H_2(F_2^*)\otimes H_1(F_3^*),\\ S_{17}=H_1(F_1^*)\otimes H_1(F_2^*)\otimes H_1(\mathrm{GL}_1), & S_{18}=H_1(F_1^*)\otimes H_1(F_3^*)\otimes H_1(\mathrm{GL}_1)\\ S_{19}=H_1(F_2^*)\otimes H_1(F_3^*)\otimes H_1(\mathrm{GL}_1), & S_{20}=H_1(F_1^*)\otimes H_1(F_2^*)\otimes H_1(F_3^*),\\ S_{21}=\mathrm{Tor}_1^{\mathbb{Z}}(H_1(F_1^*),H_1(F_2^*)), & S_{22}=\mathrm{Tor}_1^{\mathbb{Z}}(H_1(F_1^*),H_1(F_3^*)),\\ S_{23}=\mathrm{Tor}_1^{\mathbb{Z}}(H_1(F_1^*),H_1(\mathrm{GL}_1)), & S_{24}=\mathrm{Tor}_1^{\mathbb{Z}}(H_1(F_2^*),H_1(F_3^*)),\\ S_{25}=\mathrm{Tor}_1^{\mathbb{Z}}(H_1(F_2^*),H_1(\mathrm{GL}_1)), & S_{26}=\mathrm{Tor}_1^{\mathbb{Z}}(H_1(F_3^*),H_1(\mathrm{GL}_1)),\\ S'=H_3(F^*I_2\times \mathrm{GL}_2). & \end{array}$$

Note that by [8, Prop. 3.1] these decompositions are canonical. Let $x=(x_1,\dots,x_{26},x')\in\ker(\tilde{d}^1_{3,3})$. Consider $H_3(v_1)=H_3(F^{*4})=\bigoplus_{i=1}^{26}S_i$. We

have $\tilde{d}^1_{4,3}: S_{17} \to S_{17} \oplus S_{18} \oplus S_{20}$ and $\tilde{d}^1_{4,3}: S_{19} \to S_{19} \oplus S_{20}$ given by

$$\begin{array}{l} a\otimes b\otimes c \mapsto (-a\otimes b\otimes c, b\otimes c\otimes a + a\otimes c\otimes b, -a\otimes b\otimes c) \\ a\otimes b\otimes c \mapsto (a\otimes b\otimes c, -b\otimes c\otimes a - a\otimes c\otimes b - a\otimes b\otimes c). \end{array}$$

repectivcely. So we may assume $x_{17} = x_{20} = 0$. If $x_{18} = a_{18}\otimes b_{18}\otimes c_{18}$, $x_{19} = a_{19}\otimes b_{19}\otimes c_{19}$, then $0 = \tilde{d}^1_{3,3}(x) = (z_1, \dots, z_{13}) \in \bigoplus_{i=1}^{13} T_i$, where

$$z_{10} = -a_{18}\otimes b_{18}\otimes \begin{pmatrix} 1 & 0 \\ 0 & c_{18} \end{pmatrix} + a_{19}\otimes b_{19}\otimes \begin{pmatrix} 1 & 0 \\ 0 & c_{19} \end{pmatrix} = 0.$$

Thus $x_{18} = x_{19}$. Let $H_3(u_6) = H_3(F^*I_2 \times F^* \times \mathrm{GL}_1) = \bigoplus_{i=1}^{8} A_i \oplus A'$, where

$$\begin{array}{lll} A_1 = H_3(F^*I_2), & A_2 = H_1(F^*)\otimes H_2(\mathrm{GL}_1), & A_3 = H_2(F^*)\otimes H_1(\mathrm{GL}_1), \\ A_4 = H_3(F^*), & A_5 = H_1(F^*I_2)\otimes H_2(F^*), & A_6 = H_2(F^*I_2)\otimes H_1(F^*), \end{array}$$
$$A_7 = H_1(F^*I_2)\otimes H_1(F^*)\otimes H_1(\mathrm{GL}_1), \quad A_8 = \mathrm{Tor}_1^{\mathbb{Z}}(H_1(F^*), H_1(\mathrm{GL}_1)).$$

Then $\tilde{d}^1_{4,3}: A_7 \to S_{18}\oplus S_{19}$, $y \mapsto (y, y)$. So we may assume $x_{18} = x_{19} = 0$. Consider $\tilde{d}^1_{4,3}: S_5 \to S_5 \oplus S_9 \oplus S_{13} \oplus S_{15}$ and $\tilde{d}^1_{4,3}: S_6 \to S_6 \oplus S_8 \oplus S_{14} \oplus S_{16}$ given by

$$\begin{array}{l} a\otimes \sum[b|c] \mapsto (-a\otimes\sum[b|c], -\sum[b|c]\otimes a, a\otimes\sum[b|c], a\otimes\sum[b|c]) \\ \sum[d|e]\otimes f \mapsto (-\sum[d|e]\otimes f, -f\otimes\sum[d|e], \sum[d|e]\otimes f, \sum[d|e]\otimes f), \end{array}$$

respectively. Thus we may assume that $x_5 = x_6 = 0$. Now easy calculation shows that $z_4 = z_7 = x_7 + x_4' = 0$, $x_4' \in S'$. Using the map

$$\tilde{d}^1_{4,3}: A_1 \to S_4 \oplus S_7 \oplus S', y \mapsto (-y, -y, *),$$

we may assume that $x_7 = 0$. Again consider the maps

$$\begin{array}{ll} A_2\oplus A_3\oplus A_4 \xrightarrow{\tilde{d}^1_{4,3}} S_8\oplus S_9\oplus S_{10}\oplus S', & (x_8, x_9, x_{10}) \mapsto (x_8, x_9, x_{10}, *), \\ A_5\oplus A_6 \xrightarrow{\tilde{d}^1_{4,3}} S_{13}\oplus S_{14}\oplus S_{15}\oplus S_{16}\oplus S', & (x_{15}, x_{16}) \mapsto (x_{15}, x_{16}, x_{15}, x_{16}, *). \end{array}$$

So we may assume $x_8 = x_9 = x_{10} = x_{15} = x_{16} = 0$. Applying the maps

$$\begin{array}{ll} \tilde{d}^1_{4,3}: S_{25} \to S_{22}\oplus S_{24}\oplus S_{25}\oplus S_{26}, & y \mapsto (y, y, -y, -y) \\ \tilde{d}^1_{4,3}: S_{24} \to S_{21}\oplus S_{24}\oplus S_{25}, & y \mapsto (y, -y, 0) \end{array}$$

one may assume $x_{25} = x_{24} = 0$. Applying the map $\tilde{d}^1_{4,3}: A_8 \to T_{26}\oplus S'$, given by $y \mapsto (y, *)$, we may assume that $x_{26} = 0$. Let $H_3(u_4) = \bigoplus_{i=1}^{8} B_i \oplus B'$, where

$$\begin{array}{lll} B_1 = H_3(\mathrm{GL}_1), & B_2 = H_1(F^*)\otimes H_2(\mathrm{GL}_1), & B_3 = H_2(F^*)\otimes H_1(\mathrm{GL}_1), \\ B_4 = H_3(F^*), & B_5 = H_1(F^*)\otimes H_2(F^*I_2), & B_6 = H_2(F^*)\otimes H_1(F^*I_2), \end{array}$$
$$B_7 = \mathrm{Tor}_1^{\mathbb{Z}}(H_1(F^*), H_1(F^*I_2)), \quad B_8 = \mathrm{Tor}_1^{\mathbb{Z}}(H_1(F^*), H_1(\mathrm{GL}_1)),$$

Using the map $\tilde{d}^1_{4,3}: B_8 \to S_{23}\oplus S'$, $y \mapsto (y, *)$, we may assume $x_{23} = 0$. By easy calculation we have $z_8 = x_{11} - x_{13} = 0$, $z_9 = x_{12} - x_{14} = 0$ and

$z_{11} = x_{21} - x_{22} = 0$. Consider the following maps

$$\tilde{d}^1_{4,3} : B_1 \oplus B_2 \oplus B_3 \oplus B_4 \to S_1 \oplus S_2 \oplus S_3 \oplus S_4 \oplus S',$$
$$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3, x_4, *),$$
$$\tilde{d}^1_{4,3} : B_5 \oplus B_6 \to S_{11} \oplus S_{12} \oplus S_7 \oplus S_{14} \oplus S', \quad (y, y') \mapsto (y, y', y, y', *),$$
$$\tilde{d}^1_{4,3} : B_{11} \to S_{21} \oplus S_{22}, \quad y \mapsto (y, y).$$

So we may assume $x_1 = x_2 = x_3 = x_4 = x_{11} = x_{12} = x_{13} = x_{14} = x_{21} = x_{22} = 0$. This reduce x to an element of the form $x = x' \in S'$. But the map $\tilde{d}^1_{3,3} : H_3(w_2) = H_3(F^* I_2 \times \mathrm{GL}_2) \to \tilde{E}^1_{2,3} = H_3(F^{*2} \times \mathrm{GL}_2)$ is injective, thus $x = 0$. This completes the proof of the triviality of $\tilde{E}^2_{3,3}$. □

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Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany.

email: bmirzaii@math.uni-bielefeld.de